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A Decidability Theorem

The aim of this investigation is to show that supposed assumptions in the original proof of Gödel's First incompleteness theorem allow to infer a decidability of formulas that were asserted as undecidable in the theorem.

Gödel's First incompleteness theorem reads as follows¹:

Incompleteness theorem. *For every ω -consistent recursive class \varkappa of formulas there are recursive class sings r , such that neither $v \text{ Gen } r$ nor $\text{Neg}(v \text{ Gen } r)$ belongs to $\text{Flg}(\varkappa)$ (where v is the free variable of r).*

Let us express the theorem by means of the logical symbolization:

$$\forall \varkappa \left[\left(\text{recursive}(\varkappa) \ \& \ \omega\text{consist}(\varkappa) \right) \supset \exists r \left(\text{recursive}(r) \supset \overline{\left(\frac{(v \text{ Gen } r) \in \text{Flg}(\varkappa)}{\text{Neg}(v \text{ Gen } r)} \right) \in \text{Flg}(\varkappa)} \right) \right].$$

Because r does not occur as free in $\text{recursive}(\varkappa)$ and $\omega\text{consist}(\varkappa)$, given above formal expression of the theorem may be rewritten as (using exportation):

$$\forall \varkappa, \exists r \left[\left(\text{recursive}(\varkappa) \ \& \ \omega\text{consist}(\varkappa) \ \& \ \text{recursive}(r) \right) \supset \overline{\left(\frac{(v \text{ Gen } r) \in \text{Flg}(\varkappa)}{\text{Neg}(v \text{ Gen } r)} \right) \in \text{Flg}(\varkappa)} \right].$$

Three members of conjunction in the antecedent of implication in last are assumptions²:

¹ Gödel K. On formally undecidable propositions of *Principia Mathematica* and related systems I // van Heijenoort J. (ed.) Frege and Gödel: Two fundamental texts in mathematical logic. – Cambridge, Massachusetts, 1970, p. 98.

² For the first and the second it is obviously from the original proof of Gödel's First incompleteness theorem (see *Ibid.*, pp. 98–100) and also from the reconstruction of it that was made recently by author (the reconstruction is published in a volume of *Analytica*, 2009 (see URL: www.philosophy.ru/analytica, pp. 37–77 (in russian)). Concerning the third assumption note that the formula $\text{recursive}(r)$ results from $\text{recursive}(q)$ indeed (where q is a relation sign) and the last results from implicative assumption of the original proof $Q(x, y) \rightleftharpoons q(u_1, u_2)$ (see the reconstruction). But this is not significant for goals of the paper.

- 1) $recursive(\varkappa)$; 2) ${}_{\omega}consist(\varkappa)$; 3) $recursive(r)$.

These assumptions allow to derive a decidability of undecidable formulas. Before we propose the proof of this result, let us turn to six supplementary lemmas³.

Lemma 1. *If a class \varkappa of formulas is recursive, then a class $Flg(\varkappa)$ of formulas is recursive also.*

Proof. Let us prove the Lemma 1 by means of induction on length of inference.

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| 1. | $recursive(\varkappa)$ | assumption |
| 2. | $x \in Flg(\varkappa) \equiv x \in \varkappa \vee Ax(x) \vee$
$\vee (y, z \in Flg(\varkappa) \ \& \ Fl(x, y, z))$ | definition |

Basis. Let length of inference is 1. Then exist two cases (by 2): $x \in \varkappa$ and $Ax(x)$. We have:

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| 3. | $recursive(x \in \varkappa)$ | $\varkappa \equiv \{x \mid x \in \varkappa\}$, ass. 1 |
| 4. | $recursive(Ax(x))$ | Def. 42 |

Further we have:

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| 5. | $recursive(Fl(x, y, z))$ | Def. 43 |
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³ The proofs of *lemmas* and *the decidability theorem* are given with the help of the symbolization technique of metamathematical predicates that was suggested by Gödel in his system P . It is assumed that this system and definitions of following notions and relations are known by readers (see *Ibid.*):

- 1) $recursive(\varkappa) \equiv \forall a(x \in \varkappa \vee \overline{x \in \varkappa})$;
- 2) a class *sign* is a formula (a combination of signs) that has the form $a(b)$, where b is a sign of type 1 (i.e. a variable of the natural numbers) and a a sign of type 2 (i.e. a class of numbers); or has one of forms $\sim (a)$, $(a) \vee (b)$, $x\Pi(a)$, where x may be any variable;
- 3) $x \in Flg(\varkappa) \equiv x \in \varkappa \vee Ax(x) \vee (y, z \in Flg(\varkappa) \ \& \ Fl(x, y, z))$;
- 4) $x \ Gen \ y \equiv x\Pi(y)$;
- 5) $Sb(x_y^v) \equiv Subst a_b^v$;
- 6) ${}_{\omega}consist(\varkappa) \equiv \exists a (\forall n[Sb(a_{Z(n)}^v) \in Flg(\varkappa)] \ \& \ [Neg(v \ Gen \ a)] \in Flg(\varkappa))$;
- 7) $\forall R(recursive(R) \equiv decid(R))$;
- 8) $recursive(r) \equiv recursive(R) \ \& \ R \ \simeq \ r$ (where the sign ‘ \simeq ’ means a relation of one-to-one correspondence between an arbitrary relation (class) R and its isomorphic relation sign (class sign) r);
- 9) $Sb(r_{Z(n)}^v) \equiv r(Z(n))$;
- 10) $decid(R) \equiv \exists r(R \rightarrow Bew(r) \ \& \ \overline{R} \rightarrow Bew(Neg(r)))$,

and also it is known that with the help of 5th definition and some substitutions from the scheme of axiom III.1 of the system P , – i.e., $v\Pi(a) \supset Subst a_c^v$, – the axiom $v\Pi(r) \supset Sb(r_{Z(n)}^v)$ turns out.

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| 6. | $recursive(Fl(y, p, q))$ | 5 by $x/y; y/p; z/q$ |
| 7. | $recursive(Fl(z, u, w))$ | 5 by $x/z; y/u; z/w$ |
| 8. | $recursive(x \varepsilon \varkappa \vee Ax(x))$ | Theorem II ⁴ , 3, 4 |
| 9. | $recursive(y \varepsilon \varkappa \vee Ax(y))$ | 8 by x/y |
| 10. | $recursive(z \varepsilon \varkappa \vee Ax(z))$ | 8 by x/z |
| 11. | $recursive((y \varepsilon \varkappa \vee Ax(y)) \& (z \varepsilon \varkappa \vee Ax(z)))$ | Theorem II, 9, 10 |

Let length of inference is 2. Then exist four cases (by 2):

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| 1) $y \varepsilon \varkappa$ and $z \varepsilon \varkappa$ and $Fl(x, y, z)$; | 3) $Ax(y)$ and $z \varepsilon \varkappa$ and $Fl(x, y, z)$; |
| 2) $y \varepsilon \varkappa$ and $Ax(z)$ and $Fl(x, y, z)$; | 4) $Ax(y)$ and $Ax(z)$ and $Fl(x, y, z)$. |

All these cases we have in the following step:

12. $recursive[((y \varepsilon \varkappa \vee Ax(y)) \& (z \varepsilon \varkappa \vee Ax(z))) \& \text{Theorem II, 11, 5}$
 $\& Fl(x, y, z)]$

Inductive step. Let length of inference is $n + 1$. Last formula x of this inference may be 1) $x \varepsilon \varkappa$; 2) $Ax(x)$; 3) $Fl(x, y, z)$, where y and z are either $y \varepsilon \varkappa$ or $Ax(y)$ ($z \varepsilon \varkappa$ or $Ax(z)$); 4i) $Fl(x, y, z)$, where y is either $y \varepsilon \varkappa$ or $Ax(y)$; and z is such, that there are some preceding formulas u and w for that was stated $Fl(z, u, w)$; 4ii) $Fl(x, y, z)$, where y is such, that there are some preceding formulas p and q for that was stated $Fl(y, p, q)$; and z is either $z \varepsilon \varkappa$ or $Ax(z)$; 4iii) $Fl(x, y, z)$, where y and z such, that there are some preceding formulas p, q, u and w for that was stated $Fl(y, p, q)$ and $Fl(z, u, w)$. First three cases we have on lines 3, 4 and 12. Let us incorporate cases 4i, 4ii and 4iii with case on line 12. Then using inductive assumption about existence of formulas p, q, u and w we have:

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| 13. | $recursive[(y \varepsilon \varkappa \vee Ax(y) \vee$
$\vee (p, q \varepsilon Flg(\varkappa) \& Fl(y, p, q))) \&$
$\& (z \varepsilon \varkappa \vee Ax(z) \vee$
$\vee (u, w \varepsilon Flg(\varkappa) \& Fl(z, u, w))) \&$
$\& Fl(x, y, z)]$ | Theorem II, 6, 7, 9, 10 |
| 14. | $y \varepsilon Flg(\varkappa) \equiv y \varepsilon \varkappa \vee Ax(y) \vee$
$\vee (p, q \varepsilon Flg(\varkappa) \& Fl(y, p, q))$ | 2 by x/y |

⁴ Ibid., p. 93.

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| 15. | $z \in Flg(\mathcal{X}) \equiv z \in \mathcal{X} \vee Ax(z) \vee$
$\vee (u, w \in Flg(\mathcal{X}) \ \& \ Fl(z, u, w))$ | 2 by x/z |
| 16. | $recursive(y \in Flg(\mathcal{X}) \ \& \ z \in Flg(\mathcal{X}) \ \& \ Fl(x, y, z))$ | 2-change rule, 13, 14, 15 |
| 17. | $recursive(x \in \mathcal{X} \vee Ax(x) \vee$
$(y \in Flg(\mathcal{X}) \ \& \ z \in Flg(\mathcal{X}) \ \& \ Fl(x, y, z)))$ | Theorem II, 3, 4, 16 |
| 18. | $recursive(x \in Flg(\mathcal{X}))$ | change rule, 17, 2 |
| 19. | $recursive(Flg(\mathcal{X}))$ | $Flg(\mathcal{X}) \equiv$
$\{x \mid x \in Flg(\mathcal{X})\}$, 18 |
| 20. | $recursive(\mathcal{X}) \supset recursive(Flg(\mathcal{X}))$ | ass. elim., 1 |

□

Lemma 2. *If a class $Flg(\mathcal{X})$ of formulas is recursive, then for the given class sign r can be defined whether it belongs to the class $Flg(\mathcal{X})$ or not.*

Proof.

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|----|---|---|
| 1. | $recursive(\mathcal{X}) \equiv \forall r(r \in \mathcal{X} \vee \overline{r \in \mathcal{X}})$ | definition |
| 2. | $recursive(Flg(\mathcal{X})) \equiv \forall r(r \in Flg(\mathcal{X}) \vee \overline{r \in Flg(\mathcal{X})})$ | 1 by $\mathcal{X}/Flg(\mathcal{X})$ |
| 3. | $\forall r(A(r)) \equiv A(r)$ | logic rule |
| 4. | $\forall r(r \in Flg(\mathcal{X}) \vee \overline{r \in Flg(\mathcal{X})}) \equiv$
$\equiv r \in Flg(\mathcal{X}) \vee \overline{r \in Flg(\mathcal{X})}$ | 2 by $A(r)/r \in Flg(\mathcal{X}) \vee$
$\vee \overline{r \in Flg(\mathcal{X})}$ |
| 5. | $recursive(Flg(\mathcal{X})) \equiv r \in Flg(\mathcal{X}) \vee \overline{r \in Flg(\mathcal{X})}$ | transitivity of \equiv , 2, 4 |
| 6. | $recursive(Flg(\mathcal{X})) \supset r \in Flg(\mathcal{X}) \vee \overline{r \in Flg(\mathcal{X})}$ | \equiv -elim., 5 |

□

Lemma 3. (i) *If the given class sign r belongs to a class $Flg(\mathcal{X})$, then $v \text{ Gen } r$ belongs to the same class; (ii) *If the class sign r does not belong to the class $Flg(\mathcal{X})$, then $v \text{ Gen } r$ also does not belong to the same class.**

Proof.

(i)

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| 1. | $r \in Flg(\mathcal{X})$ | assumption |
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| 2. | $v \text{ Gen } r \equiv r$ | logic rule |
| 3. | $v \text{ Gen } r \varepsilon \text{ Flg}(\varkappa)$ | change rule,
1, 2 |
| 4. | $r \varepsilon \text{ Flg}(\varkappa) \supset v \text{ Gen } r \varepsilon \text{ Flg}(\varkappa)$ | \supset -enter.,
ass. elim., 1 |

(ii)

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|----|--|-------------------------------------|
| 1. | $\overline{r \varepsilon \text{ Flg}(\varkappa)}$ | assumption |
| 2. | $v \text{ Gen } r \equiv r$ | logic rule |
| 3. | $\overline{v \text{ Gen } r \varepsilon \text{ Flg}(\varkappa)}$ | change rule,
1, 2 |
| 4. | $\overline{r \varepsilon \text{ Flg}(\varkappa)} \supset \overline{v \text{ Gen } r \varepsilon \text{ Flg}(\varkappa)}$ | \supset -enter.,
ass. elim., 1 |

□

Lemma 4. *Suppose $v \text{ Gen } r$ belongs to a class $\text{Flg}(\varkappa)$. Let $\text{Sb}(r_{Z(n)}^v)$ be a formula that results from a class sign r by a substitution for its free variable v by a numeral of the number n ; then does not exist a number n , such that $\text{Sb}(r_{Z(n)}^v)$ does not belong to the class $\text{Flg}(\varkappa)$.*

Proof.

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| 1. | $v \text{ Gen } r \varepsilon \text{ Flg}(\varkappa)$ | assumption |
| 2. | $v \text{ Gen } r \equiv v\Pi(r)$ | definition |
| 3. | $v\Pi(r) \varepsilon \text{ Flg}(\varkappa)$ | change rule,
2, 1 |
| 4. | $v\Pi(r) \supset \text{Sb}(r_{Z(n)}^v)$ | axiom |
| 5. | $Ax(v\Pi(r) \supset \text{Sb}(r_{Z(n)}^v))$ | Def. 42, 38 |
| 6. | $Ax(v\Pi(r) \supset \text{Sb}(r_{Z(n)}^v)) \supset$
$Ax(v\Pi(r) \supset \text{Sb}(r_{Z(n)}^v)) \vee (v\Pi(r) \supset \text{Sb}(r_{Z(n)}^v)) \varepsilon \varkappa \vee$
$(y, z \varepsilon \text{ Flg}(\varkappa) \ \& \ Fl((v\Pi(r) \supset \text{Sb}(r_{Z(n)}^v)), y, z))$ | logic rule |
| 7. | $Ax(v\Pi(r) \supset \text{Sb}(r_{Z(n)}^v)) \vee (v\Pi(r) \supset \text{Sb}(r_{Z(n)}^v)) \varepsilon \varkappa \vee$
$(y, z \varepsilon \text{ Flg}(\varkappa) \ \& \ Fl((v\Pi(r) \supset \text{Sb}(r_{Z(n)}^v)), y, z))$ | <i>m.p.</i> , 6, 5 |
| 8. | $(v\Pi(r) \supset \text{Sb}(r_{Z(n)}^v)) \varepsilon \text{ Flg}(\varkappa) \equiv$ | definition |

- $Ax(v\Pi(r) \supset Sb(r_{Z(n)}^v)) \vee (v\Pi(r) \supset Sb(r_{Z(n)}^v)) \varepsilon \varkappa \vee$
 $(y, z \varepsilon Flg(\varkappa) \ \& \ Fl((v\Pi(r) \supset Sb(r_{Z(n)}^v)), y, z))$
9. $Ax(v\Pi(r) \supset Sb(r_{Z(n)}^v)) \vee (v\Pi(r) \supset Sb(r_{Z(n)}^v)) \varepsilon \varkappa \vee$ \equiv -elim., 8
 $(y, z \varepsilon Flg(\varkappa) \ \& \ Fl((v\Pi(r) \supset Sb(r_{Z(n)}^v)), y, z)) \supset$
 $(v\Pi(r) \supset Sb(r_{Z(n)}^v)) \varepsilon Flg(\varkappa)$
10. $(v\Pi(r) \supset Sb(r_{Z(n)}^v)) \varepsilon Flg(\varkappa)$ *m.p.*, 9, 7
11. $Fl(Sb(r_{Z(n)}^v), (v\Pi(r) \supset Sb(r_{Z(n)}^v)), v\Pi(r))$ Def. 43
12. $v\Pi(r) \varepsilon Flg(\varkappa) \ \& \ (v\Pi(r) \supset Sb(r_{Z(n)}^v)) \varepsilon Flg(\varkappa) \ \&$ $\&$ -enter., 3,
 $Fl(Sb(r_{Z(n)}^v), (v\Pi(r) \supset Sb(r_{Z(n)}^v)), v\Pi(r))$ 10, 11
13. $v\Pi(r) \varepsilon Flg(\varkappa) \ \& \ (v\Pi(r) \supset Sb(r_{Z(n)}^v)) \varepsilon Flg(\varkappa) \ \&$ logic rule
 $Fl(Sb(r_{Z(n)}^v), (v\Pi(r) \supset Sb(r_{Z(n)}^v)), v\Pi(r)) \supset$
 $\left\{ v\Pi(r) \varepsilon Flg(\varkappa) \ \& \ (v\Pi(r) \supset Sb(r_{Z(n)}^v)) \varepsilon Flg(\varkappa) \ \& \right.$
 $Fl(Sb(r_{Z(n)}^v), (v\Pi(r) \supset Sb(r_{Z(n)}^v)), v\Pi(r)) \left. \right\} \vee$
 $Sb(r_{Z(n)}^v) \varepsilon \varkappa \vee Ax(Sb(r_{Z(n)}^v))$
14. $\left\{ v\Pi(r) \varepsilon Flg(\varkappa) \ \& \ (v\Pi(r) \supset Sb(r_{Z(n)}^v)) \varepsilon Flg(\varkappa) \ \& \right.$ *m.p.*, 13, 12
 $Fl(Sb(r_{Z(n)}^v), (v\Pi(r) \supset Sb(r_{Z(n)}^v)), v\Pi(r)) \left. \right\} \vee$
 $Sb(r_{Z(n)}^v) \varepsilon \varkappa \vee Ax(Sb(r_{Z(n)}^v))$
15. $Sb(r_{Z(n)}^v) \varepsilon Flg(\varkappa) \equiv$ definition
 $\left\{ v\Pi(r) \varepsilon Flg(\varkappa) \ \& \ (v\Pi(r) \supset Sb(r_{Z(n)}^v)) \varepsilon Flg(\varkappa) \ \& \right.$
 $Fl(Sb(r_{Z(n)}^v), (v\Pi(r) \supset Sb(r_{Z(n)}^v)), v\Pi(r)) \left. \right\} \vee$
 $Sb(r_{Z(n)}^v) \varepsilon \varkappa \vee Ax(Sb(r_{Z(n)}^v))$
16. $\left\{ v\Pi(r) \varepsilon Flg(\varkappa) \ \& \ (v\Pi(r) \supset Sb(r_{Z(n)}^v)) \varepsilon Flg(\varkappa) \ \& \right.$ \equiv -elim., 15
 $Fl(Sb(r_{Z(n)}^v), (v\Pi(r) \supset Sb(r_{Z(n)}^v)), v\Pi(r)) \left. \right\} \vee$
 $Sb(r_{Z(n)}^v) \varepsilon \varkappa \vee Ax(Sb(r_{Z(n)}^v)) \supset$
 $Sb(r_{Z(n)}^v) \varepsilon Flg(\varkappa)$

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| 17. | $Sb(r_{Z(n)}^v) \in Flg(\varkappa)$ | <i>m.p.</i> , 16, 14 |
| 18. | $\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)]$ | \forall -enter., 17 |
| 19. | $\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)] \equiv \overline{\overline{\exists n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)]}}$ | logic rule |
| 20. | $\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)] \supset \overline{\overline{\exists n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)]}}$ | \equiv -elim., 19 |
| 21. | $\overline{\overline{\exists n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)]}}$ | <i>m.p.</i> , 20, 18 |
| 22. | $v \text{ Gen } r \in Flg(\varkappa) \supset \overline{\overline{\exists n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)]}}$ | \supset -enter.,
ass. elim., 1 |

□

Lemma 5. *Suppose a class \varkappa of formulas is ω -consistent. Let $Sb(r_{Z(n)}^v)$ be a formula that results from a class sign r by a substitution for its free variable v by a numeral of the number n ; then either $Neg(v \text{ Gen } r)$ does not belong to a class $Flg(\varkappa)$ or exists a number n , such that $Sb(r_{Z(n)}^v)$ does not belong to the class $Flg(\varkappa)$.*

Proof.

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| 1. | $\omega \text{ consist}(\varkappa)$ | assumption |
| 2. | $\omega \text{ consist}(\varkappa) \equiv$ | definition |
| | $\overline{\overline{\exists r \left(\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)] \ \& \ [Neg(v \text{ Gen } r)] \in Flg(\varkappa) \right)}}$ | |
| 3. | $\omega \text{ consist}(\varkappa) \supset$ | \equiv -elim., 2 |
| | $\overline{\overline{\exists r \left(\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)] \ \& \ [Neg(v \text{ Gen } r)] \in Flg(\varkappa) \right)}}$ | |
| 4. | $\exists r \left(\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)] \ \& \ [Neg(v \text{ Gen } r)] \in Flg(\varkappa) \right)$ | <i>m.p.</i> , 3, 1 |
| | $\overline{\overline{\exists r \left(\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)] \ \& \ [Neg(v \text{ Gen } r)] \in Flg(\varkappa) \right)}}$ | |
| 5. | $\exists r \left(\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)] \ \& \ [Neg(v \text{ Gen } r)] \in Flg(\varkappa) \right) \equiv$ | logic rule |
| | $\forall r \left(\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)] \ \& \ [Neg(v \text{ Gen } r)] \in Flg(\varkappa) \right)$ | |
| | $\overline{\overline{\exists r \left(\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)] \ \& \ [Neg(v \text{ Gen } r)] \in Flg(\varkappa) \right)}}$ | |
| 6. | $\exists r \left(\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)] \ \& \ [Neg(v \text{ Gen } r)] \in Flg(\varkappa) \right) \supset$ | \equiv -elim., 5 |
| | $\forall r \left(\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)] \ \& \ [Neg(v \text{ Gen } r)] \in Flg(\varkappa) \right)$ | |
| | $\overline{\overline{\forall r \left(\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)] \ \& \ [Neg(v \text{ Gen } r)] \in Flg(\varkappa) \right)}}$ | |
| 7. | $\forall r \left(\forall n [Sb(r_{Z(n)}^v) \in Flg(\varkappa)] \ \& \ [Neg(v \text{ Gen } r)] \in Flg(\varkappa) \right)$ | <i>m.p.</i> , 6, 4 |

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| 8. | $\overline{(\forall n[Sb(r_{Z(n)}^v) \varepsilon Flg(\mathcal{X})] \& [Neg(v Gen r)] \varepsilon Flg(\mathcal{X}))}$ | \forall -elim., 7 |
| 9. | $\overline{(\forall n[Sb(r_{Z(n)}^v) \varepsilon Flg(\mathcal{X})] \& [Neg(v Gen r)] \varepsilon Flg(\mathcal{X}))} \equiv \overline{\forall n[Sb(r_{Z(n)}^v) \varepsilon Flg(\mathcal{X})]} \vee \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}$ | A. de Morgan rule |
| 10. | $\overline{(\forall n[Sb(r_{Z(n)}^v) \varepsilon Flg(\mathcal{X})] \& [Neg(v Gen r)] \varepsilon Flg(\mathcal{X}))} \supset \overline{\forall n[Sb(r_{Z(n)}^v) \varepsilon Flg(\mathcal{X})]} \vee \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}$ | \equiv -elim., 9 |
| 11. | $\overline{\forall n[Sb(r_{Z(n)}^v) \varepsilon Flg(\mathcal{X})]} \vee \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}$ | <i>m.p.</i> , 10, 8 |
| 12. | $\overline{\forall n[Sb(r_{Z(n)}^v) \varepsilon Flg(\mathcal{X})]} \equiv \overline{\exists n[Sb(r_{Z(n)}^v) \varepsilon Flg(\mathcal{X})]}$ | logic rule |
| 13. | $\overline{\exists n[Sb(r_{Z(n)}^v) \varepsilon Flg(\mathcal{X})]} \vee \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}$ | change rule, 12, 11 |
| 14. | $\overline{\omega consist(\mathcal{X}) \supset \exists n[Sb(r_{Z(n)}^v) \varepsilon Flg(\mathcal{X})]} \vee \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}$ | \supset -enter.,
ass. elim., 1 |

□

Lemma 6. *Suppose a class sign r is recursive; then either $v Gen r$ or $Neg(v Gen r)$ belongs to a class $Flg(\mathcal{X})$.*

Proof.

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| 1. | $recursive(r)$ | assumption |
| 2. | $recursive(r) \equiv recursive(R) \& R \simeq r$ | definition |
| 3. | $recursive(r) \supset recursive(R) \& R \simeq r$ | \equiv -elim., 2 |
| 4. | $recursive(R) \& R \simeq r$ | <i>m.p.</i> , 3, 1 |
| 5. | $recursive(R) \& R \simeq r \supset recursive(R)$ | logic rule |
| 6. | $recursive(R) \& R \simeq r \supset R \simeq r$ | logic rule |
| 7. | $recursive(R)$ | <i>m.p.</i> , 5, 4 |
| 8. | $R \simeq r$ | <i>m.p.</i> , 6, 4 |
| 9. | $\forall R(recursive(R) \equiv decid(R))$ | definition ⁵ |
| 10. | $recursive(R) \equiv decid(R)$ | \forall -elim., 9 |
| 11. | $recursive(R) \supset decid(R)$ | \equiv -elim., 10 |

⁵ The corollary of Gödel's Theorem V (see *Ibid.*, p. 100.)

12.	$decid(R)$	<i>m.p.</i> , 11, 7
13.	$decid(R) \equiv \exists r(R \rightarrow Bew(r) \& \bar{R} \rightarrow Bew(Neg(r)))$	definition
14.	$decid(R) \supset \exists r(R \rightarrow Bew(r) \& \bar{R} \rightarrow Bew(Neg(r)))$	\equiv -elim., 13
15.	$\exists r(R \rightarrow Bew(r) \& \bar{R} \rightarrow Bew(Neg(r)))$	<i>m.p.</i> , 14, 12
16.	$R \rightarrow Bew(r) \& \bar{R} \rightarrow Bew(Neg(r))$	\exists -elim., 15, 8
17.	$Neg(R) \equiv \bar{R}$	definition
18.	$R \rightarrow Bew(r) \& Neg(R) \rightarrow Bew(Neg(r))$	change rule, 17, 16
19.	$Neg(Bew(r)) \rightarrow Neg(R) \&$ $Neg(Bew(Neg(r))) \rightarrow Neg(Neg(R))$	twice contra- position, 18
20.	$R \rightarrow Bew(r) \& Neg(R) \rightarrow Bew(Neg(r)) \&$ $Neg(Bew(r)) \rightarrow Neg(R) \&$ $Neg(Bew(Neg(r))) \rightarrow Neg(Neg(R))$	$\&$ -enter., 18, 19
21.	$Neg(Bew(Neg(r))) \rightarrow Neg(Neg(R)) \& R \rightarrow Bew(r) \&$ $Neg(Bew(r)) \rightarrow Neg(R) \& Neg(R) \rightarrow Bew(Neg(r))$	commutat. of $\&$, 20
22.	$(Neg(Bew(Neg(r))) \rightarrow Neg(Neg(R)) \& R \rightarrow Bew(r)) \&$ $(Neg(Bew(r)) \rightarrow Neg(R) \& Neg(R) \rightarrow Bew(Neg(r)))$	associativity of $\&$, 21
23.	$Neg(Neg(R)) \equiv R$	logic rule
24.	$(Neg(Bew(Neg(r))) \rightarrow R \& R \rightarrow Bew(r)) \&$ $(Neg(Bew(r)) \rightarrow Neg(R) \& Neg(R) \rightarrow Bew(Neg(r)))$	change rule, 23, 22
25.	$(Neg(Bew(Neg(r))) \rightarrow Bew(r)) \&$ $(Neg(Bew(r)) \rightarrow Bew(Neg(r)))$	twice transiti- vity of \rightarrow , 24
26.	$Neg(A) \rightarrow B \equiv A \vee B$	logic rule
27.	$Neg(Bew(Neg(r))) \rightarrow Bew(r) \equiv Bew(Neg(r)) \vee Bew(r)$	26 by <i>A/</i> <i>Bew(Neg(r))</i> , <i>B/Bew(r)</i>
28.	$Neg(Bew(r)) \rightarrow Bew(Neg(r)) \equiv Bew(r) \vee Bew(Neg(r))$	26 by <i>B/</i> <i>Bew(Neg(r))</i> , <i>A/Bew(r)</i>
29.	$(Bew(Neg(r)) \vee Bew(r)) \& (Bew(r) \vee Bew(Neg(r)))$	twice change rule, 27–28, 25

30. $(Bew(Neg(r)) \vee Bew(r)) \& (Bew(r) \vee Bew(Neg(r))) \supset$ logic rule
 $Bew(Neg(r)) \vee Bew(r)$
31. $Bew(Neg(r)) \vee Bew(r)$ *m.p.*, 30, 29
32. $v Gen r \equiv r$ logic rule
33. $Bew(Neg(v Gen r)) \vee Bew(v Gen r)$ change rule,
32, 31
34. $\forall x[Bew(x) \rightarrow Bew_{\varkappa}(x)]$ statement (8)⁶
35. $Bew(Neg(v Gen r)) \supset Bew_{\varkappa}(Neg(v Gen r))$ \forall -elim., 34
36. $Bew(v Gen r) \supset Bew_{\varkappa}(v Gen r)$ \forall -elim., 34
37. $\forall x[Bew_{\varkappa}(x) \equiv x \in Flg(\varkappa)]$ statement (7)⁷
38. $Bew_{\varkappa}(Neg(v Gen r)) \equiv Neg(v Gen r) \in Flg(\varkappa)$ \forall -elim., 37
39. $Bew_{\varkappa}(Neg(v Gen r)) \supset Neg(v Gen r) \in Flg(\varkappa)$ \equiv -elim., 38
40. $Bew_{\varkappa}(v Gen r) \equiv v Gen r \in Flg(\varkappa)$ \forall -elim., 37
41. $Bew_{\varkappa}(v Gen r) \supset v Gen r \in Flg(\varkappa)$ \equiv -elim., 40
42. $Bew(Neg(v Gen r)) \supset Neg(v Gen r) \in Flg(\varkappa)$ transitivity of
 \supset , 35, 39
43. $Bew(v Gen r) \supset v Gen r \in Flg(\varkappa)$ transitivity of
 \supset , 36, 41
44. $Bew(Neg(v Gen r)) \vee Bew(v Gen r) \&$
 $Bew(Neg(v Gen r)) \supset Neg(v Gen r) \in Flg(\varkappa) \&$ 33, 42–43
 $Bew(v Gen r) \supset v Gen r \in Flg(\varkappa)$
45. $(A \vee B \& A \supset C \& B \supset D) \supset C \vee D$ logic rule
46. $(Bew(Neg(v Gen r)) \vee Bew(v Gen r) \&$ 45 by *A*/
 $Bew(Neg(v Gen r)) \supset Neg(v Gen r) \in Flg(\varkappa) \& Bew(Neg(v Gen r)),$
 $Bew(v Gen r) \supset v Gen r \in Flg(\varkappa)) \supset$ *B*/*Bew(v Gen r)*,
 $Neg(v Gen r) \in Flg(\varkappa) \vee v Gen r \in Flg(\varkappa)$ *C*/*Neg(v Gen r) \in*
Flg(\varkappa), *D*/
v Gen r \in Flg(\varkappa)
47. $Neg(v Gen r) \in Flg(\varkappa) \vee v Gen r \in Flg(\varkappa)$ *m.p.*, 46, 44

⁶ Ibid., p. 99.

⁷ Ibid.

48. $recursive(r) \supset$ \supset -enter.,
 $Neg(v Gen r) \in Flg(\mathcal{X}) \vee v Gen r \in Flg(\mathcal{X})$ ass. elim., 1

□

We now had been approaching to the main goal of present paper. The main result about the decidability of undecidable propositions consists in following:

Decidability theorem. *For every ω -consistent recursive class \mathcal{X} of formulas, for all recursive class sings r strictly either $v Gen r$ or $Neg(v Gen r)$ belongs to $Flg(\mathcal{X})$ (where v is the free variable of r).*

Let us express the theorem symbolically:

$$\forall \mathcal{X}, \forall r \left[\left(recursive(\mathcal{X}) \ \& \ \omega consist(\mathcal{X}) \ \& \ recursive(r) \right) \supset \right. \\
\left. \left((v Gen r \in Flg(\mathcal{X}) \vee [Neg(v Gen r)] \in Flg(\mathcal{X})) \ \& \right. \right. \\
\left. \left. \overline{(v Gen r \in Flg(\mathcal{X}) \ \& \ [Neg(v Gen r)] \in Flg(\mathcal{X}))} \ \& \right. \right. \\
\left. \left. \overline{\overline{(v Gen r \in Flg(\mathcal{X}) \ \& \ [Neg(v Gen r)] \in Flg(\mathcal{X}))}} \right) \right].$$

Proof.

- | | | |
|-----|--|----------------------|
| 1. | $recursive(\mathcal{X})$ | assumption 1 |
| 2. | $\omega consist(\mathcal{X})$ | assumption 2 |
| 3. | $recursive(r)$ | assumption 3 |
| 4. | $recursive(\mathcal{X}) \supset recursive(Flg(\mathcal{X}))$ | Lemma 1 |
| 5. | $recursive(Flg(\mathcal{X}))$ | <i>m.p.</i> , 4, 1 |
| 6. | $recursive(Flg(\mathcal{X})) \supset r \in Flg(\mathcal{X}) \vee \overline{r \in Flg(\mathcal{X})}$ | Lemma 2 |
| 7. | $r \in Flg(\mathcal{X}) \vee \overline{r \in Flg(\mathcal{X})}$ | <i>m.p.</i> , 6, 5 |
| 8. | $r \in Flg(\mathcal{X})$ | assumption 4 |
| 9. | $r \in Flg(\mathcal{X}) \supset v Gen r \in Flg(\mathcal{X})$ | Lemma 3(i) |
| 10. | $v Gen r \in Flg(\mathcal{X})$ | <i>m.p.</i> , 9, 8 |
| 11. | $v Gen r \in Flg(\mathcal{X}) \supset \exists n \overline{\overline{[Sb(r_{Z(n)}^v) \in Flg(\mathcal{X})]}}$ | Lemma 4 |
| 12. | $\overline{\overline{\exists n [Sb(r_{Z(n)}^v) \in Flg(\mathcal{X})]}}$ | <i>m.p.</i> , 11, 10 |
| 13. | $\omega consist(\mathcal{X}) \supset$ | Lemma 5 |

	$\exists n[\overline{Sb(r_{Z(n)}^v \varepsilon Flg(\mathcal{X}))} \vee \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}]$	
14.	$\exists n[\overline{Sb(r_{Z(n)}^v \varepsilon Flg(\mathcal{X}))} \vee \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}]$	<i>m.p.</i> , 13, 2
15.	$\overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}$	resolution, 14, 12
16.	$v Gen r \varepsilon Flg(\mathcal{X}) \& \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}$	&-enter., 10, 15
17.	$r \varepsilon Flg(\mathcal{X}) \supset$ $v Gen r \varepsilon Flg(\mathcal{X}) \& \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}$	\supset -enter., ass. 4 elim., 8
18.	$\overline{r \varepsilon Flg(\mathcal{X})}$	assumption 5
19.	$\overline{r \varepsilon Flg(\mathcal{X})} \supset \overline{v Gen r \varepsilon Flg(\mathcal{X})}$	Lemma 3(ii)
20.	$\overline{v Gen r \varepsilon Flg(\mathcal{X})}$	<i>m.p.</i> , 19, 18
21.	$recursive(r) \supset$	Lemma 6
	$Neg(v Gen r) \varepsilon Flg(\mathcal{X}) \vee v Gen r \varepsilon Flg(\mathcal{X})$	
22.	$Neg(v Gen r) \varepsilon Flg(\mathcal{X}) \vee v Gen r \varepsilon Flg(\mathcal{X})$	<i>m.p.</i> , 21, 3
23.	$Neg(v Gen r) \varepsilon Flg(\mathcal{X})$	resolution, 22, 20
24.	$\overline{v Gen r \varepsilon Flg(\mathcal{X})} \& \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}$	&-enter., 20, 23
25.	$\overline{r \varepsilon Flg(\mathcal{X})} \supset$	\supset -enter.,
	$\overline{v Gen r \varepsilon Flg(\mathcal{X})} \& \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}$	ass. 5 elim., 18
26.	$(r \varepsilon Flg(\mathcal{X}) \vee \overline{r \varepsilon Flg(\mathcal{X})}) \&$ $(r \varepsilon Flg(\mathcal{X}) \supset$ $v Gen r \varepsilon Flg(\mathcal{X}) \& \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}) \&$ $\overline{r \varepsilon Flg(\mathcal{X})} \supset$ $\overline{v Gen r \varepsilon Flg(\mathcal{X})} \& \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}$	&-enter., 7, 17, 25
27.	$((a \vee b) \& (a \supset c) \& (b \supset d)) \supset c \vee d$	logic rule
28.	$((r \varepsilon Flg(\mathcal{X}) \vee \overline{r \varepsilon Flg(\mathcal{X})}) \&$ $(r \varepsilon Flg(\mathcal{X}) \supset$ $v Gen r \varepsilon Flg(\mathcal{X}) \& \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}) \&$ $\overline{r \varepsilon Flg(\mathcal{X})} \supset$	27 by $a/r \varepsilon Flg(\mathcal{X})$; $b/\overline{r \varepsilon Flg(\mathcal{X})}$; $c/v Gen r \varepsilon Flg(\mathcal{X}) \&$ $\& \overline{[Neg(v Gen r)] \varepsilon Flg(\mathcal{X})}$

29. $\left(\overline{v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X})} \right) \supset$ $d/v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \&$
 $\left(v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& \overline{[\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X})} \right) \vee$ $\& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X})$
 $\vee \left(\overline{v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X})} \right)$
 $\left(v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& \overline{[\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X})} \right) \vee$ *m.p.*, 28, 26
 $\vee \left(\overline{v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X})} \right)$
30. $\left(v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \vee [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X}) \&$ *distr. of \vee , distr. of $\&$,*
 $\overline{\left(v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X}) \right) \&}$ *A. de Morgan laws,*
 $\overline{\left(\overline{v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X})} \right)}$ *$\&$ -enter., 29*
31. *recursive(r)* \supset \supset -enter.,
 $\left(\left(v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \vee [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X}) \right) \&$ *ass. 3 elim., 3, 30*
 $\overline{\left(v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X}) \right) \&}$
 $\overline{\left(\overline{v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X})} \right)}$
32. $\omega \text{ consist}(\mathcal{X}) \supset \left(\text{recursive}(r) \supset$ \supset -enter.,
 $\left(\left(v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \vee [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X}) \right) \&$ *ass. 2 elim., 2, 31*
 $\overline{\left(v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X}) \right) \&}$
 $\overline{\left(\overline{v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X})} \right)}$
33. *recursive(x)* $\supset \left(\omega \text{ consist}(\mathcal{X}) \supset \left(\text{recursive}(r) \supset$ \supset -enter.,
 $\left(\left(v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \vee [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X}) \right) \&$ *ass. 1 elim., 1, 32*
 $\overline{\left(v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X}) \right) \&}$
 $\overline{\left(\overline{v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X})} \right)}$
34. $\left(\text{recursive}(\mathcal{X}) \& \omega \text{ consist}(\mathcal{X}) \& \text{recursive}(r) \right) \supset$ *importation, 33*
 $\left(\left(v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \vee [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X}) \right) \&$
 $\overline{\left(v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X}) \right) \&}$
 $\overline{\left(\overline{v \text{ Gen } r \in \text{Flg}(\mathcal{X}) \& [\text{Neg}(v \text{ Gen } r)] \in \text{Flg}(\mathcal{X})} \right)}$

$$\begin{aligned}
35. \quad & \forall \mathcal{X}, \forall r \left[\left(\text{recursive}(\mathcal{X}) \ \& \ \omega\text{consist}(\mathcal{X}) \ \& \ \text{recursive}(r) \right) \supset \quad 2\text{-}\forall\text{-enter.}, 34 \right. \\
& \left((v \text{ Gen } r \varepsilon \text{ Flg}(\mathcal{X}) \vee [\text{Neg}(v \text{ Gen } r)] \varepsilon \text{ Flg}(\mathcal{X})) \ \& \right. \\
& \left. \overline{(v \text{ Gen } r \varepsilon \text{ Flg}(\mathcal{X}) \ \& \ [\text{Neg}(v \text{ Gen } r)] \varepsilon \text{ Flg}(\mathcal{X}))} \ \& \right. \\
& \left. \overline{\overline{(v \text{ Gen } r \varepsilon \text{ Flg}(\mathcal{X}) \ \& \ [\text{Neg}(v \text{ Gen } r)] \varepsilon \text{ Flg}(\mathcal{X}))}} \right) \left. \right]
\end{aligned}$$

□

Q. E. D.