

6. The Peircian Ancestry of Boundary Methods.

"[A thorough understanding of mathematical reasoning] is the purpose for which my logical algebras were designed, but which they do not at present fulfill. The system of existential graphs is far more perfect in that respect." Peirce (4.429, 1903)

"The spatial relations of written symbols on a two dimensional writing surface can be employed in far more diverse ways than the mere following and preceding in one-dimensional time, and this facilitates the apprehension of that to which we wish to direct our attention. In fact, simple sequential ordering in no way corresponds to the diversity of of logical relations through which thoughts are interconnected." Frege (1882: 87)

Kauffman (2001) discusses how the **pa** resembles the graphical logic to which Peirce devoted much of his last 20 years. Peirce's logical graphs are planar representations of logic formulae, consisting of ovals, called *seps*, that may be nested, and atomic formulae written anywhere. (Peirce usually referred to seps as *cuts*, a term I avoid because of possible confusion with the Cut Rule of conventional logic, discussed in §5.4.) The graphical logic has but one very simple syntactic rule: seps cannot intersect. Seps and **BA** boundaries are functionally identical and share a common logical interpretation as denial.⁶⁵

Peirce devised two systems of graphical logic, the *entitative* and *existential* graphs. In the former, the blank page denotes *falsity*, so that juxtaposition denotes alternation. *LoF* (p. 5) unwittingly concurs with this entitative interpretation of the blank page. Hence the entitative graphs and the primal interpretation of the **pa** share the same semantics. Peirce developed the existential graphs (EG) at far greater length, even subtitling them "My Chef d'Oeuvre" (4.347-529, 1903), so that the entitative graphs need not detain us further. The EG are dual to the entitative graphs, in that the blank page denotes *truth* and juxtaposition denotes conjunction. The EG are of three kinds, the first of which, *alpha* (hereinafter *alpha graphs*), is isomorphic to the **pa**. The scope and power of Peirce's graphical logic did not become clear until Roberts (1973).⁶⁶ Nevertheless, that logic is a major precursor to boundary logic.⁶⁷

65. An advantage of the alpha graphs as a notation for the **BA** is that 2.1.5 would be unnecessary: any (*finite*) nesting or juxtaposition of nonintersecting seps would be a **PA** formula.

66. Roberts (1973) evolved out of his 1963 PhD thesis; for his more recent views on the alpha graphs, see Roberts (1997). Zeman's (1964) thesis likewise saw that *alpha* is isomorphic to CTV, but went beyond Roberts by proving that the *beta* graphs are isomorphic to first order logic with identity. Shin (2002) includes a thorough exposition of *alpha* and *beta*, also discussing (§§2.4, 2.5) Peirce's view of logic as semiotic.

67. By pointing out parallels between boundary logic and Peirce's logical graphs, I do not wish to suggest plagiarism. One should first keep in mind that *LoF* is a text of sorts rather than a scholarly tract. Moreover, *LoF* predates the publication of Peirce (1976), the crucial excerpt from Peirce's 1886 paper on the "sign of illation" (cf. §4.2 above). *LoF* cites Volume IV of Peirce's *Collected Papers* (Peirce 1933), which includes 115pp on the logical graphs, but this part of Peirce's *oeuvre* was dismissed or ignored until Roberts (1973). Roberts also made extensive use of Peirce's unpublished papers, not accessible to scholars before 1956, and not catalogued until 1967. Given when (the 1960s) and where (the UK) *LoF* was written, the secondary literature on Peirce's graphs, cited in Roberts, was almost nonexistent and difficult to access.

The alpha graphs are governed by six *conventions* (i.e., definitions, roughly), and four *rules of transformation*, akin to natural deduction rules. These are shown in Table 6-1, where CV n denotes the n th convention, and R n the n th rule of transformation. The third column of this Table proposes *LoF* counterparts to the rules and conventions.⁶⁸

SA stands for “sheet of assertion,” the blank surface on which graphs are to be written. SA represents all that is true of everything in the domain; hence a blank page implicitly asserts *truth*. For Roberts (1973: 32, 119), this assertion is the sole axiom of the alpha graphs; he calls it SA. CV1 and CV2 can be seen as defining “graph” and “domain”. I state CV3-CV5 using *alpha* as the object language and the \mathfrak{pa} as the metalanguage. CV3 defines conjunction; CV4, the conditional; CV5, denial. CV1-CV5 reveal that the semantics of the existential graphs are those of boundary logic under the dual interpretation of the \mathfrak{pa} . The rules of transformation **1i-3e** operationalize “step” in the context of the *alpha*. In Roberts’s terminology, a step *preserves tautologies*. For a comparison of Peirce’s logical graphs with other notations for logic, see Roberts (1973: 136).

Table 6-1. Peirce’s Existential Graphs and Boundary Logic		
<i>Name in Roberts (1973)</i>	<i>Remarks</i>	<i>LoF</i>
CV0 (4.394, 1903)	What is not forbidden is permitted. <i>This contradicts the PA Convention of Intention.</i>	
CV1	A blank surface asserts truth. Sometimes referred to as SA, the “sole axiom” of the alpha graphs.	T1 Dual inter.
CV2	A graph asserts some truth about the domain.	T2
CV3	$ab \Leftrightarrow a \wedge b$.	Dual inter.
CV4	$\boxed{a \boxed{b}} \Leftrightarrow (a(b))$.	“
CV5	$\boxed{a} \Leftrightarrow (a)$; $\square \Leftrightarrow () \Leftrightarrow \text{false}$.	“
1i. Insert Odd	Any subgraph may be written in an odd depth.	6.0.1
1e. Erase Even	Any evenly enclosed subgraph may be erased.	“
2i. Iteration	$a(b) \rightarrow a(ab)$; $a \rightarrow aa$.	T13, C5
2e. Deiteration	$a(ab) \rightarrow a(b)$; $aa \rightarrow a$.	“
3i. Insert double cut	$a \rightarrow ((a))$; write $(())$ anywhere	C1, A2
3e. Delete double cut	$((a)) \rightarrow a$; erase any instance of $(())$.	“

2i-e is a joint consequence of T13 and C5; **3i-e** is C1 and A2 in a new guise. **1i-3e**, proved in Roberts (1973: §3.2), can also be proved in the \mathfrak{pa} . **1i-e** are unfamiliar because they are peculiar to the dual interpretation. First note that if the inserted/erased (sub)formula evaluates to \perp , **1i-e** follow trivially from A2. More generally, we have:

6.0.1. Theorem. **1i, 1e** preserve tautologies.

68. *LoF* includes a few diagrams in the spirit of Peirce’s graphs; see chapter 12, the notes thereto, and p. 115.

Proof. Let γ be a subformula of the formula α . It follows from T14 that there exists a formula $\beta = \alpha$, also containing γ , such that the depth of γ in β does not exceed 2. Alternatively, if γ is to be inserted in α , the result is equivalent to inserting γ in some β whose depth also does not exceed 2. Hence only three cases need be considered: γ has depth 0, 1, or 2. The EG map very naturally into the dual interpretation of the \mathbf{pa} , so that the objective is to reduce β to \perp .

Erase γ at depth 0. *Cal.* $a\gamma b' \rightarrow ab' \Leftrightarrow (a\gamma b'(ab')) = ((ab')ab\gamma') [OI] = \perp [J1]$.

Insert γ at depth 1. *Cal.* $(ab') \rightarrow (a\gamma b') \Leftrightarrow ((ab')((a\gamma b'))) = ((ab')ab'\gamma) [C1; OI] = \perp [J1]$.

Erase γ at depth 2. *Cal.* $(a(b\gamma)) \rightarrow (ab') \Leftrightarrow ((a(b\gamma))((ab'))) = ((\underline{a}(b\gamma))ab') [C1] = (((b\gamma))ab') [C2] = (b\gamma ab') [C1] = (b'b\gamma a) [OI] = \perp [J1]$.

The depth=0,2 cases justify *Erase Even*; the case depth=1, *Insert Odd*.

I now show that the three remaining possibilities do not reduce to tautologies:

Insert γ at depth 0. *Dem.* $ab' \rightarrow a\gamma b' \Leftrightarrow (ab'(a\gamma b')) = (ab'\gamma') [OI; C2, 2x]$.

Erase γ at depth 1. *Dem.* $(a\gamma b') \rightarrow (ab') \Leftrightarrow ((a\gamma b')((ab'))) = ((\underline{a\gamma b'})ab') [C1] = (\gamma' ab') [C2, 2x]$.

Insert γ at depth 2. *Dem.* $(ab') \rightarrow (a(b\gamma)) \Leftrightarrow ((ab')((a(b\gamma)))) = ((\underline{ab'})a(b\gamma)) [C1] = ((\underline{b'})a(b\gamma)) [C2] = (ba(b\gamma)) [C1] = (ba\gamma') [C2]. \square$

Remark. The six demonstrations making up 6.0.1 require only C1 (**3i-e**), C2 (**2i-e**), OI (tacit in the EG), and J1. Hence J1 plays the same role in BA that **1i, 1e** play in EG.

Showing that the \mathbf{pa} can in turn be derived from *alpha* merely requires demonstrating J1 in *alpha*, to wit:

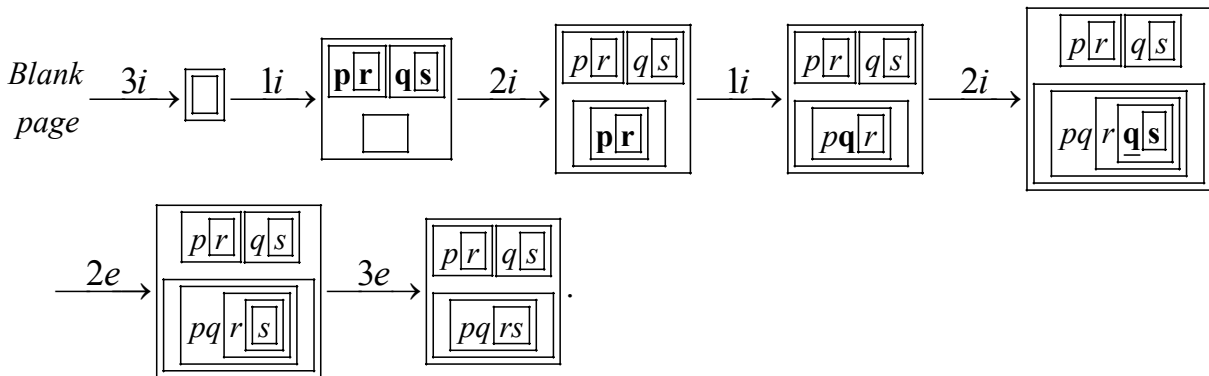
Blank

$$\text{page} \xrightarrow{3i} \boxed{\quad} \xrightarrow{1i} \boxed{\mathbf{a}\quad} \xrightarrow{2i} \boxed{a\mathbf{a}}.$$

(SA)

Hence **1i-3e** form a basis for the \mathbf{pa} . There is a sense in which J1 does the work of **1i-e**. I submit, however, that J1 is more intuitive than **1i-e**, and eliminates any need to keep track of depth parity.

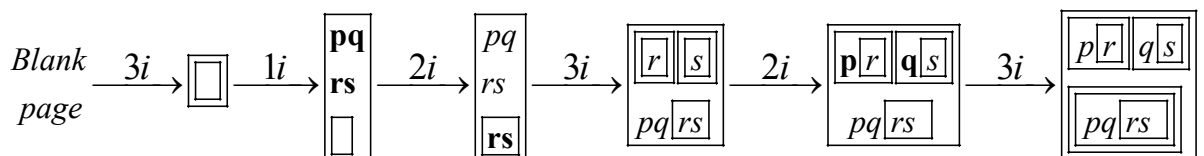
I now give an example of the power of the alpha graphs, Sowa's (2002) demonstration of Leibniz's *Praeclarum Theorema*, $[(p \rightarrow r) \wedge (q \rightarrow s)] \rightarrow [(p \wedge q) \rightarrow (r \wedge s)]$, verified by TVA in §5.3. Variables inserted by **1i** or duplicated by **2i** first appear in bold; variable instances eliminated by **2e** are underlined.



This demonstration is a vast improvement over that of *PM* (*3.47) which, Sowa tells us, requires 43 steps, starting from five axioms. I now verify the *Theorema* via a *pa* calculation. In keeping with the spirit of the *alpha* graphs, the *pa* translation invokes the dual interpretation.

$$\text{Cal. } ((pr')(qs')((pq(rs)))) \text{ [translation]} = ((\underline{pr}')(\underline{qs}')pq(rs)) \text{ [C1]} = ((r')(s')pq(rs)) \text{ [C2,2x]} \\ = (pqrs(rs)) \text{ [C1,2x; OI]} = \perp \text{ [J1]}.$$

Reading this calculation in reverse suggests the following *alpha* demonstration:



The last demonstration requires but a single insertion, requiring no reflection: insert at the outset, in any order, one instance of each of the four variables appearing in the *Theorema*. Sowa’s demonstration, on the other hand, invokes **1i** twice, each time requiring careful thought about what to insert. Having demonstrated the *Theorema* via TVA, *alpha* graphs (twice), and a *pa* calculation, I invite the reader to decide which demonstration is the most perspicuous.

6.1. Other Systems Related to the *pa*.

Ba’s radically “arithmetical” foundation for Boolean algebra and the truth functors has a curious precedent. Van Horn (1917), writing in ignorance of Sheffer (1913), purported to derive *PM*’s CTV axioms from a single axiom governing the semantics of the Sheffer stroke (which I will notate by \uparrow), restated as follows: $[|a|=|b|] \rightarrow [|a\uparrow b| \neq |a|]$, and $[|a| \neq |b|] \rightarrow [|a\uparrow b| = \top]$. This axiom follows trivially from A1, A2, keeping in mind that $[a\uparrow b \Leftrightarrow (a)(b)] \Leftrightarrow [() \Leftrightarrow \top]$ and $[a\uparrow b \Leftrightarrow (ab)] \Leftrightarrow [() \Leftrightarrow \top]$. The axiom also follows from 4.1.1 above. Nicod (1917: 40) praised Van Horn’s paper, but also claimed that its derivations of the *PM* axioms were flawed. I note that Van Horn’s proofs invoke what is in effect T16 of the *pa*, without any awareness that he was invoking a metatheorem needing proof. Even though Van Horn’s and Nicod’s papers were published side by side, Grattan-Guiness (2001: 434) is the only citation of Van Horn I know of. While it appears that no one ever revisited Van Horn’s paper, I submit that the *pa* shows how its intuition was valid.⁶⁹

69. The way *LoF* grounds **Ba** in a bit of Boolean arithmetic is not without precedent and parallels. Shannon (1938) gives the following basis for **2**: (a) his arithmetical postulates (1)-(3), isomorphic to Table 2-1a; (b) his assumption (4) that *B* has two members; and (c) a loose definition of complementation. Table 2-1b is isomorphic to his “theorems” (7a) and (7b). Prior (1962: 4-13), following Polish practice, grounds his exposition of sentential logic in the Boolean arithmetic of 0 and 1. Cole (1968) derives **Ba** from analogues of Table 2-1, R1, and T16. Malmstadt et al (1973: 281, §3-4.1) derive **2**, sketchily, from Boolean addition (dually, multiplication) and complementation, taken as axiomatic. Rudeanu (1974) asserts (Example 1.1), but does not show, that his axioms for **Ba** (namely OI, absorption, Boolean versions of J1, J2, C3, and the duals of all the preceding, as per his Definition 1.1) can, in the case of **2**, “be easily established by direct verification” from the operation tables for Boolean addition,

Byrne's (1946) **Ba** notation is based on juxtaposition (deliberately uninterpreted) as sole connective and the unary prime. It translates into the **pa** as follows: (1) remove parentheses not immediately followed by a prime; (2) if a pair of parentheses is primed, remove the prime; (3) replace subformulae of the form $\alpha\alpha'$ by $()$. Byrne's notation diverges from that of this paper only in the use it makes of parentheses. Curiously, six of Byrne's eight theorems are C1-C6. Byrne proves his algebra Boolean by deriving Huntington's (1933a) basis, C6 and OI. Byrne has been oft-cited but to my knowledge, his notation goes unimitated. Likewise, the boundary mathematics literature, such as it is, does not seem aware of Byrne's work.

The system AB of Anderson and Belnap (1959) features one unary operation, denial, denoted by an overbar, and one connective, disjunction, denoted by \vee . Erasing all instances of \vee , and enclosing in parentheses that which lies under an overbar, results in the equivalent **pa** formula. Evidently, AB is semantically identical to the primal interpretation of the **pa**.⁷⁰ The sole axiom of AB is $b(a)ac = ()$, true by J0. The rules of inference of AB are (1) from bac , infer $b((a)c)$, true by C1, and (2) from $(a)c$ and $(b)c$, infer $(ab)c$. *Cal.* $(a'\underline{c})(b'\underline{c})(ab)c = (a')\underline{(b')}(ab)c$ [C2,2x] = $\underline{ab(ab)}c$ [C1,2x] = $()$ [J0]. \square

AB explicitly disavows *modus ponens*, as its authors were well-known advocates of non-truth functional strict implication, and is silent about substitution as a rule of inference. Hunter (1971: §37.6) sets out an effective proof procedure for AB based on refutation trees, but the resulting proofs are a good deal more complicated than ones based on J0-C2. For example, Hunter requires 16 lines to verify that $(p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$. The corresponding **pa** calculation is trivial: *Cal.* $(\underline{p'}\underline{q'}\underline{r})(\underline{p'}\underline{q})\underline{p'}r = (\underline{q'})\underline{(q)}\underline{p'}r$ [C2,3x] = $()$ [J0]. \square

Angell (1960) proposed a notation for first order logic and set membership whose truth functional part was $(\phi) \Leftrightarrow \neg\phi$ and $(\phi\phi) \Leftrightarrow \phi \wedge \phi$, with ϕ and ϕ being metalogical notation for formulae. The outer parentheses of $(\phi\phi)$ are needed only because of peculiarities of Angell's notation for quantification and membership. In contexts devoid of these, the outer parentheses are dispensable, in which case Angell's notation for the truth functors becomes identical to the **pa**. Semantically speaking, Angell's notation is equivalent to the dual interpretation of the **pa**. The only proper symbols of Angell's system are '(' and ')'. Hence Angell unwittingly showed that there are semantics that render the **PA** adequate for set theory and hence all standard mathematics. Angell supplied no axioms or proof theory for his notation, because he only wished to show it capable of expressing Quine's (1951) version of first order logic.

Translated into **BA**, Schütte's (1977: 17) axioms for sentential logic are I: $(aa')=\perp$ [J1], and II: $(\perp)=()$ [A2]. His rules $I\wedge$, $I\vee$, and $I\rightarrow$ are all instances of J0. It is unclear where in Schütte's system an equivalent to C2 resides.

multiplication, and negation. (Shannon 1938 makes a similar assertion when discussing his (1a)-(8).) Most of Rudeanu's axioms are redundant, in that J1, J2, and OI suffice as a **pa** basis, the **pa** is a **Ba** by 3.3.4, and is complete by T17.

70. To Hunter (1971: §37) I owe my discovery of the system AB. In a striking feat of bravado, Anderson and Belnap (1959) lay out AB and prove it sound, complete, decidable, and axiom independent in less than 300 words. Thus AB enjoys the dubious distinction of being the tersest version of sentential logic. Hunter (§§37.1-5) expands the proofs of these metatheorems to all of three whole pages.

Boundary logic is a classical logic simply because $J0$ holds is a BA tautology. Intuitionist logic, for present purposes, denies $J0$ and $C1$ and does not permit the interdefinability of \wedge , \vee , and \rightarrow by means of negation. Now recast CTV statements whose sole connective is \rightarrow in the following boundary notation. Single letters are atomic formulae. If a, b, c are formulae, then ab , $(ab)c$, and $a(bc)$ are formulae. The semantics are $ab \Leftrightarrow a \rightarrow b$. The axioms $(ab)b = (ba)a$ [alternation commutes] and $a(bc) = b(ac)$ [Sheffer stroke commutes] result in a *positive implication algebra*, which models all tautologies whose sole connective is the conditional and that intuitionists deem valid. Add $(ab)a = a$ [C4] to obtain *implication algebra* (IA; Abbott 1969: §7-4; Wolfram 2002: 803). IA stands to the classical implicational calculus, consisting of all classically valid tautologies whose sole connective is the conditional, as BA stands to the CTV. **2** is IA plus the axiom $\perp a = bb$ [C3].

7. Why the Indifference?

"[The **pa**] is a very beautiful version of the propositional calculus, and I cannot understand why it has not become a standard method in logic text-books... Spencer-Brown's theory has gained great popularity among various people, but logicians have taken little interest in it." (Grattan-Guinness 1982: §5.1).

Notwithstanding the merits I and others have claimed for it, the **pa** remains unadopted. To explore the reasons behind this fate, I begin by setting out what this curious book reveals about its origins. Spencer-Brown worked out many of the ideas in *LoF* while teaching an introductory course in logic (pp. xii, 112; page references without citation are to *LoF*); *LoF* may have begun as lecture notes for this course. He derived his version of boundary algebra (namely, one using ' \perp ' instead of ' $()$ ', and with no analogue to ' \perp ') by working backwards from **2** and CTV, discovering some key insights while designing electronic circuits during the 1960s. He claimed significant engineering experience but did not cite Shannon's (1938) celebrated result that the algebra of switching circuits is a model for the CTV (and hence also for the **pa**).

LoF indulges in philosophical speculation (pp. v, vi, xix-xxii, 85, 89-96, 101-106) and repeatedly invokes dubious etymology (pp. 93, 101, 105, 106, 109, 126). Spencer-Brown claims (p. ii) to have studied under Wittgenstein⁷¹ (whom he cites four times) and R D Laing, but is silent on how he learned mathematics and logic. (Elsewhere, he claims to have worked with Lord Cherwell in the 1950s and the mathematician J C P Miller in the 1960s.) He rightly cites some classics: Huntington (1904, 1933), Post (1921), and Sheffer (1913), as well as Boole, Peirce, Peirce's student Ladd-Franklin, and *PM*. *LoF* cites no text on Boolean algebra extant when it was written (e.g., the 1958 ed. of Hohn 1966; Whitesitt 1961; Arnold 1962; Goodstein 1963). Hence *LoF* does not disclose the extent to which it builds on Boolean algebra.

Spencer-Brown's understanding of the CTV and its terminology is not always standard. For instance, he believes (p. 115ff) that a formula true under some but not all assignments of truth values to its sentential variables has a "contingent" truth value, when the preferred designation for such a formula is *satisfiable*. The only formal logic text he cites

71. Whose spectre haunts *LoF*, as well as much of British philosophy of the mid-20th century. I leave to others the pleasure of tracing the specific influence of Wittgenstein's *oeuvre*, the *Tractatus* in particular, on *LoF*. A similar pleasure undoubtedly awaits the Peirce or Whitehead expert willing to give *LoF* a close reading.

is Prior (1962), invoked near the end of *LoF* to make a minor point about syllogisms. Logic texts extant at the time *LoF* was written include Hilbert and Ackermann (1950), the 1950 ed. of Quine (1982), Rosenbloom (1950), Quine (1951), Rosser (1953), Church (1956), Suppes (1957), Carnap (1958), Nidditch (1962), and Kneebone (1963). Finally, *LoF* appears unaware of nonclassical logics, and oblivious to the role of quantification in the first order logic consensus.

Spencer-Brown argues (chpt. 11) that certain infinite *pa* formulae with a finite recursive representation have an “imaginary” truth value, arising in a manner analogous to the way complex numbers arise from the roots of polynomial equations interpreted recursively.⁷² He alleges that such truth values have momentous mathematical and practical implications; this provocative contention remains to be verified. For example, he asserts, without elaboration, that imaginary truth values render the well-known limitative theorems of Gödel and Church (Stoll 1963: §§9.9, 9.10; Mendelson 1997: Ths. 3.37-3.54) “...less destructive than was hitherto supposed” (*LoF*, p. xvii). Anyone doubting Gödel’s classic theorem should consult the simple proof in Hersh (1997: 311-16). Spencer-Brown also writes: “[Recursive Boolean] equations have hitherto been excluded from the subject matter of ordinary logic by the Russell-Whitehead theory of types” (p. xviii).⁷³ The Theory of Types is meaningful only in a context that includes predicate logic, and *LoF* is innocent of such as well as of all but the most naïve set theory (the Boolean algebra of classes makes a very casual appearance in Appendix II).

While *LoF* does not claim that *BA* suffices to ground all of mathematics, others stride boldly where angels fear to tread: “...the propositional calculus...develops naturally from [A1 and A2]. Thus the act of severance leads inexorably to logic and through *PM* to the whole of mathematics.” (Croskin 1978: 187) This statement would be true if boundary algebra could model the quantification and second order logic *PM* requires, and if *PM* had succeeded in its aims (on why it did not see, e.g., Quine 1941).

Spencer-Brown is especially guilty of falsified predictions about the future course of mathematics. Writing in 1967, he claimed that:

“...if we confine our reasoning to an interpretation of Boolean equations of the first degree only, we should *expect* to find theorems which will always

72. If a Boolean equation has a *pa* representation that is not recursive, *LoF* (p. 57) says that is “of the first degree”. Recursive equations are said to be of “degree higher than one.” If no member of *B* solves an equation of higher degree, then *LoF* (pp. 58, viii-x) argues that the equation has an “imaginary” solution. Relating this imaginary truth value to extant work on recursive arithmetic and functions (e.g., Mendelson 1997: chpts. 3,5; Kneebone 1963: chpt. 10) and on multi-valued logic (Urquhart, 2001)—*LoF* mentions neither subject—are possible directions for future research.

73. Before touching on Russell’s paradox and the like, Spencer-Brown would have benefited from giving Prior (1962: §III.3.3) a close reading. He cites the 1958 edition of Fraenkel, Bar-Hillel, and Levy (1973: chpt. III) in support of his contention (p. xix, fn 8) that there have been prior attempts to “...rehabilitate, on a logical rather than on a mathematical basis, something of what was discarded with the Theory of Types...”. This is a very curious reading indeed. Both editions survey type-theoretical approaches to set theory, but never maintain that such approaches are or ever were standard. Re type-theoretic set theory, see also Mendelson (1997: 290-93).

defy decision, and the fact that we do seem to find such theorems in common arithmetic may serve, here, as a practical confirmation of this obvious prediction. To confirm it theoretically, we need only to prove (1) that such theorems *cannot* be decided by reasoning of the first degree, and (2) that they *can* be decided by reasoning of a higher degree. (2) would of course be proved by providing such a proof of one of the theorems.

“I may say that I believe that at least one such theorem will shortly be decided by the methods outlined in [*LoF*]. In other words, I believe that I have reduced their decision to a technical problem which is well within the capacity of an ordinary mathematician who is prepared, and who has the patronage or other means, to undertake the labour.” (pp. 99-100; emphasis in original)

More pointedly:

“...I found evidence, in unpublished work undertaken in 1962-65, suggesting that the four-colour map theorem [N.B: *a conjecture at the time of writing*] and Goldbach’s conjecture are undecidable with a proof structure confined to Boolean equations of the first degree, but decidable if we are prepared to avail ourselves of equations of higher degree.” (p. xix)⁷⁴

Here Spencer-Brown asserts that certain very well-known mathematical conjectures cannot be proved using standard mathematics grounded on classical bivalent logic, but could be proved using mathematics grounded in the 3-valued logic (i.e., one incorporating an imaginary truth value) he introduces in *LoF*. A third of a century has elapsed since *LoF* appeared and nothing of this sort has eventuated. Instead, seven years after *LoF* appeared in print, Haken and Appel announced their proof of the Four Colour Map Theorem, one based on conventional mathematics albeit supplemented by a large amount of machine computation (for a definitive treatment see Haken and Appel 1989).

Regarding Fermat’s Last Theorem (FLT), Spencer-Brown wrote:

“...it is my guess that Fermat (who was apparently too excellent a mathematician to make a false claim to a proof) used [imaginary truth values] in the proof of his great theorem, hence the ‘truly remarkable’ nature of his proof, as well as its length.” (p. 99).

Wiles (1995) finally proved FLT using difficult albeit conventional mathematics. All this may explain why Spencer-Brown’s work has been ignored, and why it need not be swallowed whole.⁷⁵

8. Conclusion.

“Logical laws are the most central and crucial statements of our conceptual scheme, and for this reason the most protected from revision by the forces of conservatism; but... they are the laws an

74. Spencer-Brown repeated his claim of a proof of the Four Colour Map theorem in a letter to the editor of *Nature*, dated 17.12.76.

75. Lest I seem too critical, I hasten to add that *LoF* is an instance of a worthy genre, namely introductions to formal logic intended for nonspecialists. A fine instance of the genre is Hodges (1977), built around refutation trees and linguistic examples.

apt revision of which might offer the most sweeping simplification of our whole system of knowledge." Quine (1982: 3)

Let there be a blank surface, Peirce's "Sheet of Assertion" (SA). On SA, make a mark. The mark can be the $\bar{\quad}$ of *LoF* (proposed by Peirce in 1886), any simple closed curve (proposed by Peirce from 1896 onwards), or $()$, Croskin's (1978) notation, advocated here. The symbol $()$ is the sole primitive constant, one that is both operator and operand. Whatever the mark is taken to be, it is essential that it have a distinguishable "interior" and "exterior," as the mark serves as the boundary between this interior and exterior. Interpreting SA as either 'true' or 'false' gives rise to boundary logic.

The only means we have, at this stage, to distinguish anything is to write another mark on the state we wish to distinguish. To mark the exterior [interior], write $()()$ [$((()))$]. The Law of Calling says that $()()$ cannot be distinguished from simple $()$. The Law of Crossing says that $((()))$ cannot be distinguished from the SA. Hence the exterior and interior of a mark are distinguished simply by the way each interacts with another mark. The exterior [interior] of a mark is idempotent [nilpotent]. The Laws of Calling and Crossing, written as $()()=()$ and " $\perp [=((()))]$ can be written or erased at will," are the sole axioms of the primary arithmetic, PA.

LoF shows, in a rather cryptic fashion, that these notions, plus logical equivalence, an equivalence relation, yields the PA. The PA can be interpreted as a Boolean arithmetic written so as to highlight its tree structure. The set $B=\{(),\perp\}$ corresponds to the carrier of **2**. For a brief recapitulation of the PA, see 2.3.4.

Inserting letters anywhere in a PA formula yields a **pa** (primary algebra) formula. A letter, commonly called a variable, can assume any value of B . Add the rules of Substitution and Replacement to obtain the **pa**. Combine the PA and **pa** to obtain boundary algebra (BA). If $()$ interprets Boolean 0 or 1, the **pa** [PA] is isomorphic to Boolean algebra [arithmetic]. BA and boundary logic are equational rather than ponential, i.e., they privilege tautological equivalence over mere tautology.

An initial is a tautological equivalence verified by a decision procedure. Any set of CTV axioms, or Boolean algebra basis, translates into a set of **pa** initials. The initials $a'a=()$ and $(ba)a=b'a$, and the well-known consequences $aa=a$, $(a')=a$, and $(a'b')r=((ar)(br))$ enable *calculation*, a style of demonstration similar to, but easier than, that of Peirce's *alpha* existential graphs. The **pa** facilitates clausal reasoning, and trivializes the derivations of the inference rules of conventional logic. The **pa** and the CTV share a common metatheory.

The **pa** is, at minimum, a simple yet powerful notation for the truth functors and Boolean algebra, revealing the unity and simplicity underlying the seeming diversity of truth functors. But the **pa** is more than a mere improved notation; **pa** calculations are much easier than the corresponding proofs taught in standard texts, especially natural deduction proofs. Moreover, because the CTV and the Boolean algebra **2** are models of the **pa**, BA highlights the seldom mentioned axiomatic role of Boolean arithmetic for these systems. I further conjecture that BA is Boolean algebra free of the ontological commitments of set theory, but leave the exploration thereof for another day.

Logic and abstract algebra are significant human inventions. To date, however, these inventions have had a modest impact on our wider civilisation, electrical engineering and computer science excepted. As a demotic version of the hieratic languages **2** and the

CTV, the **pa** could be taught in secondary schools: pragmatically, as an introduction to the abstractions underlying information technology; intellectually, as a gentle introduction to logic and discrete math. (Hehner 2000 likewise advocates teaching his Binary Algebra to children.) Thus **BA** could help make logic and algebra better known. Boundary algebra also suggests that mathematical logic, set theory, theoretical computer science, and probability⁷⁶ ultimately all share a common source: the mental act of making a distinction, for which the marker is the boundary sign $()$.

In Misner et al's (1973: 1208-12) classic text on gravitation and relativity, John Wheeler (citing Kneebone 1963, as his reference for logic) speculates that the physics of space-time will ultimately be grounded in the CTV. More recently, Brukner and Zeilinger (2002) propose to recast quantum mechanics in terms of information theory. Should either of these research programs prove fruitful, **BA** could come to play a role in theoretical physics.

In a less speculative vein, the **BA** enables a unified treatment of Boolean algebra, the CTV, and monadic logic. This treatment should include a restatement of **BA** conforming to current terminology and norms of rigor, and dispensing with the enigmatic "canons" of *LoF*. The ontological grounding and semantics of the **PA** need firming, and the boundary analogue to refutation trees should be explored. Boundary methods should prove fruitful for lattice theory, the grounding for nonclassical (e.g., intuitionist, modal, multi-valued, relevant) logics, and may shed new light on algebraic and combinatory logic, and on algebraic structures generally.⁷⁷

76. A Boolean foundation for discrete probability results from taking mathematical expectation as primitive, then defining probability as the expectation of a Boolean random variable. See Lad (1996: §2.2).

77. Meguire (2004) includes boundary representations of quantification, set theory, groupoids, lattices, and modal logic. On connections between Boolean and other algebras, see Rudeanu (1974: §§12.3-7) and Burris et al (1981: §II.1). For a survey of algebraic logic, see Andréka et al (2001). For an introduction to combinatory logic, see Rosenbloom (1950: §3.4). On non-classical propositional logic, see Restall (2000) and Epstein (1995).

Bibliographic Postscript.

Boundary algebra lies at the intersection of four disciplines: mathematics, philosophy, computer science, and electrical engineering. The references below I found useful, or they were extant and relevant at the time *LoF* was written. I group the references by broad topic, and list them in order of increasing perceived difficulty.

- There are two distinct perspectives on *Boolean algebra*:
 - *Mathematical*: Arnold (1962: §§2,5), Hohn (1966: §§5.1-5), Goodstein (1963: §§2,3), Halmos and Givant* (1998: §§19-39), Stoll* (1963: §6), Rosenbloom* (1950: §§I.1-3, II.3), Abbott (1969: §§6,7), Cori & Lascar* (2000: §2), Koppelberg (1989), Burris et al (1981: §§II.1, IV.1-4). Algebras more general than **2** are typically assumed, and are developed in either a *set theoretic* or *algebraic* manner. The starred references discuss the close connection between Boolean algebra and CTV.
 - *Engineering/Computer Science*: Whitesitt (1961: §§1-3), Hohn (1966: §1), Rudeanu (1974: esp. §1; many references).
- *Lattice theory*. Arnold (1962: §§3,4), Donnellan (1968), Curry (1963, §4; especially relevant for logic), Davey and Priestley (2002), Burris et al (1981: §§I,II.1).
- *Calculus of Truth Values*. *Mathematical*: Arnold (1962: §1), Hohn (1966: §§3.1-12), Goodstein (1963: §4), Kneebone (1963: §§2,6), Halmos and Givant (1998: §§8-18), Epstein (1995: §§II.J-M), Stoll (1974: §§2.1-5,3.5), Nidditch (1962), Machover (1996: §7), Mendelson (1997: §1), Hunter (1971: §§15-36), Smullyan (1968: Part I), Cori & Lascar (2000: §1), Curry (1963: §§5,6), Schütte (1977: §I). *Philosophical*: Girle (2002: Part One), Quine (1982: Part I), Suppes (1957: §§1,2), Bostock (1997: §2), Prior (1962: 1-71, 301-6, 318-19), Hodges (2001: §§1-7), Carnap (1958: §§2-8, 12a, 22), Zeman (1973: 1-76), Segerberg (1982).
- *Calculus of Quantified Individuals*. *Mathematical*: Stoll* (1974: §2.6-9, §3.6), Machover* (1996: §8), Quine* (1951: §2), Hunter (1971: §§38-59), Pollock (1990: §2.1), Smullyan (1968: Part II), Mendelson* (1997: §2), Schütte (1977: §II), Cori & Lascar (2000: §§3,4). *: Includes axiomatic set theory. *Philosophical*: Girle (2002: §§12-14), Quine (1982: Parts II, III), Bostock (1997: §§3,5), Hodges (2001: §§8-18), Carnap (1958: §§1, 9-14, 21-25). Bostock, Hodges, and Machover are best for current terminology.

Wolf (1998: Unit 1) introduces logic as a part of elementary mathematics. Stoll (1974: §3) is a relaxed introduction to metamathematics and axiomatic thinking. Kneebone (1963) and DeLong (1971) are gentle introductions to the philosophy of mathematics.

Boolean algebra was systematized by Schröder in the 1890s, and came of age as serious mathematics thanks to Marshall Stone, Tarski, and others in the 1930s. MacColl (Frege) invented the CTV (FOL) in 1877 (1879). Mathematical logic came into its own with *Principia Mathematica*, and the work of Hilbert and his students between the Wars. On the history of logic and of related mathematics, see Curry (1963), Kneebone (1963), Grattan-Guinness (2001), and “Logic, History of” in the *Encyclopedia Britannica*.

There is a fair secondary literature on *LoF*; a bibliography can be found at <http://www.lawsofform.org/bib/index.html>, where the names Bricken, Kauffman, and Varela stand out, as do articles in the *International Journal of General Systems*. Another *LoF*-related URL is www.enolagaia.com/GSB.html. Both sites reveal that *LoF*'s love of paradox and enigma have attracted a nonmathematical following.

Appendix: Demonstrations, Proofs, etc.

Throughout this Appendix, 'OI' means that order irrelevance is invoked. The symbol '□' signals the end of a proof/demonstration/calculation.

A.1. The Core LoF Consequences.

The demonstrations of C6 and C7 are from *LoF*. The other demonstrations below differ from those in *LoF* mainly in that the initials are taken to be J0,C2. The calculations of C5, C1, and J2 are original; the demonstration of J2 is adapted from that in Bricken (1986). Demonstrating C9 requires C6, calculating it requires C6 and C7. Otherwise all steps below require only one of J0-C2.

J1. *Dem.* $((a)a) = (())$ [J0]. □

C5. *Cal.* $aa=a$. *LR:* $(\underline{a})a = (a)a$ [C2] = $(())$ [J0]. *RL:* $(a)aa = (aa)aa$ [C2] = $(())$ [J0]. □

C1. *Cal.* $((a))=a$. *LR:* $((a))a = (((a)\underline{a})a)a$ [C2,2x] = $(a)a$ [J1] = $(())$ [J0]. *RL:* $(a)((a)) = (())$ [OI; J0]. □

J2. *Dem.* $\mathbf{a((b)(c))} = a((ab)(ac))$ [C2,3x] = $(a')\chi$ [C1; $\chi=((ab)(ac))$] = $(a'\chi)\chi$ [C2] = $(a'(a'\underline{ab})(a'\underline{ac}))\chi$ [C2,3x] = $(a'(a'))\chi$ [J1,2x] = χ [J1] = $\mathbf{((ab)(ac))}$. □

J2. *Cal.* $\mathbf{a((b)(c))} = \mathbf{((ab)(ac))}$.

LR: $(a(b'c'))((ab)(ac)) = (a(\underline{ab})(\underline{ac}))((ab)(ac))$ [C2,2x] = $(a)((ab)(ac))$ [C2] = $a'(a'(\underline{a'ab})(\underline{a'ac}))$ [C2,3x] = $a'(a')$ [J1,2x] = $(())$ [J0].

RL: $(\underline{((ab)(ac))})a(b'c') = (\underline{ab})(\underline{ac})a(b'c')$ [C1] = $ab'c'(b'c')$ [C2,2x; OI] = $(())$ [J0]. □

C3. *Dem.* $(\mathbf{a})a = (\mathbf{a})\underline{aa}$ [J0] = $(aa)aa$ [C2] = (\mathbf{a}) [J0]. □

C4. *Dem.* $(\mathbf{a'b})a = (a'ab)a$ [C2; OI] = \mathbf{a} [J1]. □

C6. *Dem.* $(\mathbf{a'b'})\mathbf{(a'b)} = (((a'b')(\underline{a'b})))$ [C1] = $(a'(\underline{(b'b')}))$ [J2] = (a') [J1] = \mathbf{a} [C1]. □

C7. *Dem.* $(\mathbf{(a'b)c}) = (((a)((b)))c)$ [C1] = $(\underline{((ac)(b'c))})$ [J2] = $\mathbf{(ac)(b'c)}$ [C1]. □

C9 here is simpler than its *LoF* counterpart. Hence the following demonstration is new:

C9. *Dem.* $\mathbf{((b'r)(a'r'))} = (\underline{((b'r)a((b'r)r)})$ [J2] = $((b'r)a)((b'r))$ [C1; C2] = $(a(b'r))(br)$ [C1; OI] = $(a(b'r)(br))(br)$ [C2] = $(a(\underline{(r'b')((r'b))})(br)$ [OI; C1, 2x] = $\mathbf{(ar')(br)}$ [C6, r'/A, b/B]. □

Normally, it is easier to calculate a consequence than demonstrate it. C9 is an exception; the *RL* part of the following calculation is surprisingly difficult and requires C7.

Cal. *LR:* $(\underline{((b'r)(a'r'))})(a'r')(br) = (rb')(rb)(r'a')(r'a)$ [C1; OI] = $r'r$ [C6,2x] = $(())$ [J0].

RL: $((a'r')(br))((b'r)(a'r')) = (r(b'r))(a'(br))(b(a'r'))(r'(a'r'))$ [C7,2x] = $(r(b))(a'(br))(b(a'r'))(r'(a'))$ [C2,2x] = $(rb')(a'(br))(b(a'r'))(r'a)$ [C1] = $(rb')(a'(\underline{rb'})\underline{(rb)})(b(\underline{r'a})(\underline{r'a}))(r'a)$ [C2,2x] = $(rb')(a'r')(br)(r'a)$ [C6,2x] = $(rb)(rb')(r'a)(r'a)$ [OI] = $r'r$ [C6,2x] = $(())$ [J0]. □

The following TVA verifies the *RL* half of the calculation more easily:

$r=()$: $((a)(b()))((b'())(a')) = ((a))((a'))$ [C3; A2] = aa' [C1,2x] = $(())$ [J0].

$r=\perp$: $((a())(b))((b')(a'())) = ((b))((b'))$ [C3; A2] = bb' [C1,2x] = $(())$ [J0].

Absorption. *Dem.* $a(a'b') = (a'b')a$ [OI] = $((ab')b')a$ [C2] = $(\underline{(ab')ab'})a$ [C2] = a [J1]. Dually, $(a'(ab)) = (a'(a'ab))$ [C2] = $(a'(\underline{(ab)ab}))$ [C2] = (a') [J1] = a [C1]. □ The lattice theory version of absorption, $a(ab)=a$, follows if $a(ab) \Leftrightarrow a(a'b')$, then $(a'(ab))$.

A.2. Demonstrating the Ba Consistency Principle.

Theorem 3.3.5. $a \leq b$, $a \cup b = b$, and $a \cap b = a$ are equivalent Ba statements.

Dem:

$$a \cup b = b \Leftrightarrow ab = b \Leftrightarrow ((ab)(b))(abb) = ((\underline{b'ba})b')(abb) \text{ [C2; OI]} = \underline{(b')}(abb) \text{ [J1]} = b(abb) \text{ [C1]} = b(a) \text{ [C2,2x]} \Leftrightarrow a \leq b.$$

$$a \cap b = a \Leftrightarrow (a'b') = a \Leftrightarrow (((a'b')\underline{a}')(a'b')a) = (a'b'a)((a'b')a) \text{ [C1]} = (a'b'a)((\underline{a'ab'})a) \text{ [C2; OI]} = (\underline{a'b'a})(a) \text{ [J1]} = \underline{(b')}(a) \text{ [C2,2x]} = b(a) \text{ [C1]} \Leftrightarrow a \leq b. \square$$

A.3. OI and C5 Suffice to Show that Juxtaposition Is Order Irrelevant.

Demonstrating that juxtaposition commutes and associates, while contrary to the planar spirit of the pa, is nevertheless possible. First note that setting $c = \perp$ in OI yields TR: $ab = ba$. Now let OI be read, here only, as $((ab))c = ((bc))a$. OI justifies each step, unless otherwise noted. Then:

Associativity. *Dem.* $((ab))c = ((bc))a = a((bc))$ [TR]. \square

An alternative demonstration of TR, adapted from Byrne (1946: 271), requiring only OI and C5 and no resort to the PA, goes as follows. Note that the calculation of C5 in A.1 does not require OI.

Commutativity. *Dem.* $ab = ((ab))ab$ [C5] = $((b((ab))))a = (((((ab))a))b) = (((((ba))a))b) = (((((aa))b))b) = ((bb))((aa)) = ba$ [C5,2x]. \square

Associativity can also be demonstrated from the initials J0, J2, and TR, but the details are tedious; hence the following is but a sketch. Eves (1990: 217-19) supplies the details, starting from Huntington's (1904) axioms J1, J2, $a \perp = a$, and TR (cf. Table 5-2).

Sketch of the Demonstration. As usual, ab [(a'b')] replaces $a \cup b$ [$a \cap b$] in Huntington and Eves, so that a single pa equation does duty for a dual pair of axioms. Huntington's third axiom is a pa consequence: *Dem:* $a \perp = a(a'a)$ [J1] = $(a'a)a$ [TR] = $(a')a$ [C2] = aa [C1] = a [C5]. \square Note that C1 and C2 are required. Eves's demonstration requires C3, C5, and Absorption. Any demonstration in which three or more subformulae never appear concatenated does not require associativity. Inspection confirms that J0 and J2 satisfy this condition. Moreover, the demonstration in A.1 of Absorption and J1, and the LoF demonstrations of C1, C2, C3, and C5 from J1 and J2, also satisfy this stipulation. Hence associativity is demonstrable from J0, J2, and TR.

A.4. Demonstrating J0 and C2 from C6 and OI.

Huntington (1933) showed that C5, C6, and OI form a basis for Ba. Huntington (1933a) showed C5 redundant by deriving it from C6 and OI. The following simpler demonstration that C6 and OI form a basis, adapted from Kauffman (1990)⁷⁸, also implies that Johnson's axioms C1 and C5 are redundant. LoF (p. 88) cites (1933) but not (1933a). I now derive J0 and C2 from OI and C6.

J0. *Dem.* $a'a = (ab)(ab')(a'b)(a'b')$ [C6,2x] = $(ab)(a'b)(ab')(a'b')$ [OI] = $(ba)(ba')(b'a)(b'a')$ [OI] = $b'b$ [C6,2x]. Hence the value of $a'a$ is the same for all a , so that $a'a$ can be equated to

78. See <http://www.rgshoup.com/lof/logic.html>. This URL also includes proofs from Bricken (1986).

either primitive value. I go with $a'a=()$, but $a'a=\perp$ is equally valid and gives rise to the dual interpretation. J1 is demonstrable by a strictly parallel reasoning. \square

C1. Dem. $(a') = (((a'))a')(((a'))(a'))$ [C6] = $(((a'))a')((a')a')$ [J1, (a')/A, a'/B] = **a** [C6]. \square

C5. Dem. $aa = ((aa))$ [C1] = $((a(a))(aa))$ [J1] = $((((a))a)((a)a))$ [C1,2x] = $((a))$ [C6] = **a** [C1]. \square

C2. Dem. $a'b = (ab)(\underline{ab})(b'a)(b'a')$ [C6,2x] = $(ab)(b'a)(b'a')$ [OI; C5] = **(ab)b** [C6]. \square

A.5. $\alpha=\beta \Leftrightarrow \alpha\leftrightarrow\beta \Leftrightarrow ((\alpha'\beta)(\beta'\alpha))$ Because ' \leftrightarrow ' Is an Equivalence/Congruence Relation.

The following calculations employ ' \Leftrightarrow ' in place of ' $=$ ', because I am temporarily suspending belief in ' $=$ ' as an equivalence relation.

Symmetric: $[\alpha=\beta] \Leftrightarrow [\beta=\alpha] \Leftrightarrow [\alpha\leftrightarrow\beta] \Leftrightarrow [\beta\leftrightarrow\alpha]$.

LR: $((\alpha'\beta)(\beta'\alpha))((\beta'\alpha)(\alpha'\beta)) \Leftrightarrow (\alpha'\beta)(\beta'\alpha)((\beta'\alpha)(\alpha'\beta))$ [C1] $\Leftrightarrow ()$ [C2,2x].

RL: Also evaluates to (). \square

Transitive: Let χ stand for $(\alpha'\beta)(\beta'\alpha)(\beta'\delta)(\delta'\beta)$. ' $\alpha=\beta$ and $\beta=\delta$ implies $\alpha=\delta$ ' translates as $(\alpha'\beta)(\beta'\alpha)(\beta'\delta)(\delta'\beta)((\alpha'\delta)(\delta'\alpha)) \Leftrightarrow \chi((\alpha'\delta)(\delta'\alpha)) \Leftrightarrow ((\chi\alpha'\delta)(\chi\delta'\alpha))$ [J2].

Now evaluate $\chi\alpha'\delta \Leftrightarrow (\underline{\alpha'}\beta)(\beta'\alpha)(\beta'\delta)(\delta'\beta)\alpha'\delta \Leftrightarrow (\beta)(\beta'\alpha)(\underline{\beta'})\delta(\delta'\beta)\alpha'\delta$ [C2,2x] $\Leftrightarrow \underline{\beta'}(\beta'\alpha)\underline{\beta}(\delta'\beta)\alpha'\delta$ [C1] $\Leftrightarrow ()(\beta'\alpha)(\delta'\beta)\alpha'\delta$ [OI; J0] $\Leftrightarrow ()$ [C3]. $\chi\delta'\alpha \Leftrightarrow ()$ by a strictly parallel reasoning. \square

Reflexive: $[\alpha=\beta] \Leftrightarrow ((\alpha'\alpha)(\alpha'\alpha)) \Leftrightarrow ()$ [J1,2x]. \square

By virtue of satisfying conditions C₁ and C₂ below, ' $=$ ' is also a *congruence relation* of the sort Boolean algebra tacitly requires (Stoll 1963: 259-61). I now demonstrate this fact, $\forall a,b,c \in B$, again translating ' $=$ ' as ' \leftrightarrow ':

C₁. $a=b \rightarrow ac=bc$. *Cal.* $(a'b)(b'a)((\underline{ac})bc)((bc)ac) \Leftrightarrow (a'b)(b'a)((\underline{a}b\underline{c})(\underline{b}a\underline{c}))$ [C2,2x] $\Leftrightarrow (a'b)(b'a)((a'b)(b'a))c$ [J2] $\Leftrightarrow ((a'b)(b'a))(a'b)(b'a)c$ [OI] $\Leftrightarrow ()$ [J0].

C₂. $a=b \rightarrow a'=b'$. *Cal.* $(a'b)(b'a)((\underline{a'}b')(\underline{b'}a')) \Leftrightarrow (a'b)(b'a)((ab')(ba'))$ [C1,2x] $\Leftrightarrow ((a'b)(b'a))(a'b)(b'a)$ [OI] $\Leftrightarrow ()$ [J0]. \square

C₃. $a'=b' \rightarrow a=b$. *Cal.* $((\underline{a'}b')(\underline{b'}a'))((a'b)(b'a)) \Leftrightarrow (ab')(ba')((ab')(ba'))$ [C1,2x] $\Leftrightarrow ((a'b)(b'a))(a'b)(b'a)$ [OI] $\Leftrightarrow ()$ [J0]. \square

Remark. Stoll does not mention C₃, which enables replacing C₂ with $a'=b' \leftrightarrow a=b$, a corollary of which being that J0 and J1 are equivalent. The calculations for C₁, C₂, and C₃ reveal that they share a common structure at the point where J0 is invoked. That ' $=$ ' is a congruence relation enables a proof of R1 (Stoll 1963: 260).

A.6. More on C2.

First a quick TVA proof. Given $(ba)a = (b)a$, let:

- $b=\perp$: Simply erase b . The lhs becomes $(a)a = ()$ [J0]. Rhs: $()a = ()$ [C3].
- $b=()$: The lhs simplifies as $(())a = (())a$ [C3]. Rhs: simply $(())a$.

Invoke T7 twice to complete the proof. \square

Equivalents of C2 include Johnson's (1892: 342) Law of Exclusion, the third consequence proved in his system; Th. 7 in Byrne (1946), (48) in Rosser (1953: 113), T8-6.j4 in Carnap (1958), T12 in Stoll (1963: 257), and exercise 3.12.a in Hohn (1966). C2 is a trivial corollary of $(b \rightarrow a) \leftrightarrow [(b \vee a) \leftrightarrow a]$, *4.72 in *PM*, T73 in Kalish et al (1980: §II.11), and (38) in Cori and Lascar (2000: 32). C2 also follows trivially from (31) in Suppes (1957: 204), once it is understood that Suppes's notation $a \sim b$ translates $(a'b)$. C2 can be obtained from

$[(a \rightarrow b) \wedge (c \rightarrow b)] \leftrightarrow [(a \vee c) \rightarrow b]$, *4.77 in *PM*, (18) in Stoll (1974: 85), and (57) in Cori and Lascar (2000: 33), via b/c and noting that $[(a \rightarrow b) \wedge (b \rightarrow b)] \leftrightarrow (a \rightarrow b)$.

One half of the biconditional in C2 is C2': $(b \rightarrow a) \rightarrow [(b \vee a) \rightarrow a]$, *2.621 in *PM* (the other half is *2.67), 2.20 in Zeman (1973), and Example 21 in Leblanc and Wisdom (1976: 99). Zeman shows that C2' is merely a substitution instance of a tautology equivalent to *modus ponens*. The converse of C2' is likewise a substitution instance of his 2.12. Zeman derives C2' from that part of the implicational calculus intuitionists accept. C2' is also an axiom in a system Hilbert set out in 1922 (system 1.3) and in four other CTV axiom systems set out in Epstein (1995: 408-9). A trivial substitution turns Reichenbach's (1947: 39) (8h)⁷⁹, $((b \vee c) \rightarrow a) \rightarrow (b \rightarrow a)$, into the converse of C2'. Even though C2' is a substitution instance of Mendelson's (1997: 35) axiom A3, his proof of the converse (his Th. 1.11.g) requires 43 lines and the Deduction Theorem! C2' can be viewed as an analogue to the special case $\phi = \Delta$ of the "left" version of Bostock's (1997: §2.5) structural rule THIN.

Another related tautology is C2'', $(b \rightarrow a) \rightarrow [(b \vee c) \rightarrow (a \vee c)]$, axiom *1.6 of *PM* (system 6.11). Hence C2'' is included in the *PM* axiom system as modified by Lukasiewicz and Bernays (Table 5-2; system 6.11 net of (4)) and commonly used since (e.g., Carnap 1958: 86, P1-P4; Kneebone 1963: 43; Mendelson 1997: 45, system L1; Halmos and Givant 1998: 22, T1-T4). To obtain C2', substitute a for c in C2'' and note that this axiom system trivially implies $(a \vee a) \leftrightarrow a$, C5 in the *pa*. As best as I can determine, however, the only proof in the sources I cite that invoke any of C2, C2', or the converse of C2' is that of Zeman's 2.21. As a result, extant expositions of the CTV are, I submit, unnecessarily complicated.

A.7. J1-C5 Are Standard Features of Logic and Boolean Algebra.

Table A-1.		
	<i>Algebra of Propositions</i>	BA
1	Idempotent	C5
2	Associative	Tacit, but provable
3	Commutative	Tacit
4	Distributive	J2
5	Identity	A2
6	"	C3
7b	Complement	J1
8a	"	C1
8b	"	A2
9	De Morgan's	Trivial, by transcription

Table A-1 shows the correspondence between Lipschutz's (1964: 195) "Laws of the Algebra of Propositions" and BA. Each Law is actually a dual pair, 8a and 8b excepted; 8a is self-dual. Corresponding to each law is a law for the algebra of sets (p. 104). When *LoF* was written, Lipschutz was a standard elementary treatment of set theory and sentential logic. Only two basic BA notions are missing from Table A-1: A1, a trivial consequence of 5 and 6, and C2.

79. *Cal.* $((b \vee c) \rightarrow a) \rightarrow (b \rightarrow a) \Leftrightarrow ((bc) \underline{a}) b' a = ((bc) \underline{a}) b' a$ [C2] = $bc b' a$ [C1] = () [OI; J0]. \square

A.8. The pa metatheorems.

T14. Let $d_\alpha^* > 2$ for some formula α . Then α can be transformed, by taking steps, into an equivalent formula β such that $d_\beta^* = 2$.

Proof. (The repeated use of C7 is from *LoF*; the proof is otherwise new.) α can be seen as an *ordered tree*, one or more of whose *branches* terminate at some maximal depth d_α^* . Let $d_\alpha^* > 2$, and let β , χ , and ϕ be subformulae appearing at depths d_α^* , $d_\alpha^* - 1$, and $d_\alpha^* - 2$, respectively, of any longest branch of α . Let γ denote all of α not accounted for by β , χ , and ϕ , so that $\alpha = (((\beta)\chi)\phi)\gamma$. By C7, $(((\beta)\chi)\phi)\gamma = (\beta\phi)(\chi'\phi)\gamma$. The maximum depth of $(\beta\phi)(\chi'\phi)\gamma$ is 1 less than that of $(((\beta)\chi)\phi)\gamma$. This depth-reducing procedure based on C7 can be repeated, each time suitably redefining β , χ , ϕ , and γ . When one branch of α is exhausted, switch to the longest remaining branch. Continue until no branch of α has depth > 2 . \square

Remark: For more on ordered trees, see Smullyan (1968: §§I.0-1). Each application of C7 to the *terminus* of any branch of α reduces that branch's depth by 1. This fact enables the following, perhaps simpler, algorithm. Begin at any branch, applying C7 repeatedly until a *node* is encountered, then switch to the terminus of any other branch. Continue until no further applications of C7 are possible, at which time no branch will have depth > 2 .

T15. Let the pa formula $\alpha\langle v \rangle$ contain more than 2 instances of the variable v . Then α can be transformed, by taking steps, into an equivalent formula $\beta\langle v \rangle$, such that $\beta\langle v \rangle$ contains at most 2 instances of v .

Proof (Adapted from *LoF*). By C1 and T14, there exist a subformula $f\langle v \rangle$, and the sequences of subformulae, a_i , p_i , and x_j , such that

$$\begin{aligned}
 \alpha\langle v \rangle &= ((va_i)p_i)\dots f(vx_i)\dots \\
 &= (((v')(a'_i))p_i)\dots f(vx_i)\dots && \text{Apply C1 twice for each value of } i. \\
 &= (\underline{v'}\underline{p}_i)(\underline{a}'_i\underline{p}_i)\dots f(vx_i)\dots && \text{Apply J2 and C1 once for each value of } i. \\
 &= (v'p_i)\dots (a'_ip_i)\dots f(vx_i)\dots && \text{By OI, the disjuncts } (v'p_i) \text{ can grouped together, to the} \\
 & && \text{left of the } (a'_ip_i). \\
 &= (v'p_i)\dots g(vx_j)\dots && \text{Let } g = (a'_ip_i)\dots f. \\
 &= (((\underline{v'}\underline{p}_i)\dots))((\underline{v}x_j)\dots))g && \text{C1, 2x; OI } g. \\
 &= ((p'_i\dots)v')((x'_j\dots)v)g && \text{T10, 2x. } \square
 \end{aligned}$$

T16. Let v be a possible value of the variable v appearing in at least one of formulae α and β . Let $\alpha\langle v \rangle$ be $\alpha\langle v \rangle$ with v set to v . If $\forall v[\alpha\langle v \rangle = \beta\langle v \rangle]$, then $\alpha = \beta$.

Proof (Adapted from *LoF*, pp. 47-49). Let $v \in B$, with one or both of $\alpha\langle v \rangle$ and $\beta\langle v \rangle$ being the case. Let v vary between the limits of its domain, \perp and $()$. Now consider the following two mutually exclusive and exhaustive cases:

1. Variation in v either alters or does not alter *both* $\alpha\langle v \rangle$ and $\beta\langle v \rangle$. If $\alpha\langle v \rangle = \beta\langle v \rangle$ after a change in v , then $\alpha\langle v \rangle = \beta\langle v \rangle$ must have held before the change. Hence $\alpha = \beta$ in this case.
2. Variation in v causes the value of one of $\alpha\langle v \rangle$ or $\beta\langle v \rangle$ to change, but *not both*. If this were so, selecting a suitable value of v would result in $\alpha\langle v \rangle \neq \beta\langle v \rangle$, contrary to hypothesis. Hence if varying the value of v causes one of $\alpha\langle v \rangle$ or $\beta\langle v \rangle$ to change, the other must change as well, so that the reasoning of case 1 applies.

Hence $\alpha\langle v \rangle$ and $\beta\langle v \rangle$ are equivalent in all cases of v , so that $\alpha=\beta$ in any case. \square

Remark. If only one of $\alpha\langle v \rangle$ and $\beta\langle v \rangle$ is the case, then variation in v cannot affect the value of the formula in which it appears.

T17. The **pa** is complete.

Proof. (Adapted from *LoF*, pp. 50-52) Let α_k, β_k be formulae each containing some variable v . Let $k-1$ be the number of distinct variables other than v appearing in either α_k or β_k . The proof proceeds by strong induction on k , with the inductive hypothesis being $\alpha_k=\beta_k \forall k < n$, for some integer $n > 0$. T15 assures us that there exist formulae α_n^* and β_n^* , with v appearing at most twice in each, such that $\alpha_n^* = \alpha_n$ and $\beta_n^* = \beta_n$. Moreover, T14 assures us that α_n^* and β_n^* can be expressed in such a way that their respective depths do not exceed 2. Since α_n^* and β_n^* are normal forms of α_n and β_n , they can be written as:

$$(E1) \quad \alpha_n^* = (v'a_1)(va_2)a_3 \qquad (E2) \quad \beta_n^* = (v'b_1)(vb_2)b_3,$$

where a_i and $b_i, i=1,2,3$, are suitable sub-formulae whose depth do not exceed 1.

Now let $v=()$ and $v=\perp$ in turn, with each case resulting in formulae for α_{n-1}^* and β_{n-1}^* :

$$\begin{aligned} v=(): \quad \alpha_{n-1}^* &= (((())a_1)((a_2)a_3) = (a_1)a_3; & \beta_{n-1}^* &= (((())b_1)((b_2)b_3) = (b_1)b_3. \\ v=\perp: \quad \alpha_{n-1}^* &= ((\perp)a_1)(\perp a_2)a_3 = (a_1)a_3; & \beta_{n-1}^* &= ((\perp)b_1)(\perp b_2)b_3 = (b_2)b_3. \end{aligned}$$

The inductive hypothesis asserts that $\alpha_{n-1}^* = \beta_{n-1}^*$ can be proved. Hence:

$$\begin{aligned} (E3) \quad \alpha_{n-1}^* &= \beta_{n-1}^* \rightarrow (a_1)a_3 = (b_1)b_3, \text{ and} \\ (E4) \quad \alpha_{n-1}^* &= \beta_{n-1}^* \rightarrow (a_2)a_3 = (b_2)b_3, \end{aligned}$$

are provable. I now demonstrate that $\alpha_n^* = \beta_n^*$. N.B.: C9: $((a'r')(b'r)) = (ar')(br)$.

$$\begin{array}{ll} (v'a_1)(va_2)a_3 & E1, \text{ transcription of } \alpha_n^*. \\ ((v'(a_1))(v(a_2)))a_3 & C9, a_1/A, a_2/B, v/R \\ ((v'(a_1)a_3)(v(a_2)a_3)) & J2, a_3/R \\ ((v'(b_1)b_3)(v(b_2)b_3)) & E3, E4 \\ ((v'(b_1))(v(b_2)))b_3 & J2, b_3/R \\ (v'b_1)(vb_2)b_3 & C9, b_1/A, b_2/B, v/R \\ \beta_n^* & E2. \end{array}$$

All that remains to be shown is that there exists a value of n for which the inductive hypothesis, $\alpha_k=\beta_k \forall k < n$, holds. Suppose that n to be 1, in which case only $k=0$ need be considered. Now if $k=0$, then α_0 and β_0 are PA formulae. $\alpha_0=\beta_0$ can be proved in the **pa** if A1 and A2 are **pa** consequences, which I proceed to show: A1 is C5, $()/A$. A2 is C1, \perp/A , and *Dem.* $\perp\perp=\perp$ [C5, \perp/A]; $(\perp)=((\perp))=()$ [C1, $()/A$]; $\perp()=()\perp=()$ [C3, \perp/A]. \square

T17. Alternate Proof (following Kneebone 1963: 48).

I take completeness to mean: *all tautologies can be demonstrated from the pa initials*. By T14, any formula α has an NF representation $\beta=(a_i^* \dots)_j \dots$. Keep in mind that for some j, i may equal 1. Every step in the derivation of β from α is justified by invoking one of C7, C1, or C2, all initials or consequences derivable from the initials. Hence if β can be

shown derivable from the initials, then α is derivable as well. Also if β is a tautology, α is as well. To demonstrate that β is a tautology, there are two cases to consider:

Case 1. $\forall j(a_i^* \dots)_j = \perp$. If every disjunct contains some variable in both primed and unprimed form, then $(a_i^* \dots)_j = \perp$ is demonstrable by applying J1 to each disjunct, resulting in $\beta = \perp$.

Case 2. $\exists j(a_i^* \dots)_j = ()$. This is demonstrable if there exists a variable x such that one disjunct is simply (x) and another is $((x))$. Then $\beta = ()$ by J0.

J0, C1, C2, and C7 are either initials, or can be derived from the initials. Hence if α is a tautology, the initials suffice to verify that fact. \square

T18. The initials J1 and J2 are independent.

Proof. (Adapted from *LoF*). J1: $(a'a) = \perp$. J2: $((ar)(br)) = (a'b')r$.

In J1, no variable instance crosses a boundary. In J2, r moves from depth 2 to depth 0. Hence J2 cannot be proved using J1 alone. Meanwhile, J1 creates or eliminates the variable a . J2 does not create or eliminate any variable. Hence J1 cannot be proved using J2 alone. \square

T18a. The initials J0, C2, and OI are independent.

Proof. J0 creates or eliminates a variable; C2 and OI do not. C2 creates or eliminates some but not all instances of a variable. J0 [OI] creates and destroys all [no] instances of a variable. OI has no boundaries while J0 and C2 each have one boundary. Hence J0, C2, and OI are mutually independent. \square

A.9. Proof of Theorem 4.1.3.

This is Bostock's (1997: §2.10) Third Duality Theorem, and Quine's (1982: §12) 5th law of duality. The proof is by *induction on formula length*, a standard technique well-explained in Bostock (1997: §2.8). The proof follows Bostock closely, except for notation.

Definition: Given a formula α , its *length* $l(\alpha) =$ number of variables in α less 1, plus the number of left parentheses and primes in α . If $l(\beta) < l(\alpha)$, then β is *shorter* than α .

Notation: Given the **pa** formula $\alpha = \alpha \langle x_1, \dots, x_n \rangle$, its *contradual* is $\bar{\alpha} =_{\text{df}} \alpha \langle x'_1, \dots, x'_n \rangle$.

Lemma (Bostock's First Duality Theorem; Quine's second law): $\alpha^D = (\bar{\alpha})$.

Proof. The hypothesis of strong induction is: For all formulae β such that $l(\beta) < l(\alpha)$, $\beta^D = (\bar{\beta})$.

There are three cases to consider, each verified by calculation:

Case 1. α is atomic. $\alpha^D = \alpha = ((\alpha))$ [C1] = $(\bar{\alpha})$ [Def. of overbar].

Case 2. $\alpha = (\beta)$.

$$\begin{aligned} \alpha^D &= (\beta)^D && \text{Substitution} \\ &= (\beta^D) && \text{Def. of duality} \\ &= ((\bar{\beta})) && \text{Inductive hypothesis} \\ &= ((\underline{\beta})) && \text{Def. of overbar} \\ &= (\alpha) && \text{Substitute } \alpha \text{ for } (\beta). \end{aligned}$$

Case 3. $\alpha = \beta\chi$ for some formula χ .

$\alpha^D = [\beta\chi]^D$	Substitution
$= ((\beta^D)(\chi^D))$	Def. of duality
$= (((\bar{\beta}))(\bar{\chi}))$	Inductive hypothesis
$= (\bar{\beta\chi})$	C1, 2x
$= (\beta\bar{\chi})$	Def. of overbar
$= (\alpha)$	Substitute α for $\beta\chi$. \square

Theorem 4.1.3: $\alpha = \phi$ iff $\alpha^D = \phi^D$.

Proof. If $\alpha = \phi$, then $\bar{\alpha} = \bar{\phi}$ [R2], so that $(\bar{\alpha}) = (\bar{\phi})$. Hence $\alpha^D = \phi^D$ by the above lemma. \square

Glossary of Mathematical Logic.

This Glossary reviews some notions from *First Order Logic (FOL)* = {Calculus of Truth Values} \cup {Calculus of Quantified Individuals}. For a masterly précis of FOL and its extension to axiomatic set theory, see Fraenkel, Bar-Hillel, and Levy (1973: §V.2); also see Hodges (2001: §§IV.A,C). For more leisurely expositions, consult the references cited under “Quantifier Logic” in the Bibliographic Postscript), the *Encyclopedia Britannica* articles “Logic” and “Metalogic,” and <http://plato.stanford.edu/entries/logic-classical/>.

A *string* consists of a single symbol, or of concatenated symbols. *Symbol* is undefined. $B = \{T, F\}$ is the set of possible *truth or primitive values*.

Calculus of Truth Values, CTV. A *statement (sentence, proposition)* is a string that can be assigned a truth value. Statements include *formulae*, i.e., strings that satisfy a formation rule. The definitions of *subformula* and *atomic formula* in 2.1.4 carry over to FOL in the obvious way. A *statement letter (sentential variable)* stands for any member of some set of statements. An n -ary ($n \in \mathbb{N}$) truth functor (aka *operator* (if $n \geq 1$) or *connective* ($n \geq 2$)) is a mapping from B^n onto B . The constants T and F are 0-ary functors by assumption. Common functors include the prefix \neg “not”, and the infix connectives \wedge “and”, \vee “or”, \rightarrow “if”, \leftrightarrow “iff”, $|$ “NAND”.

An *atomic valuation* (i) assigns an element of B to every atomic formula, and (ii) completely describes the mapping $f_k: B^n \rightarrow B$ for every n -ary operator k . A statement consisting of $m \in \mathbb{N}$ statement variables (*arguments*) and constants, linked by connectives, is a *truth function* from B^m onto B . The *truth value* of a statement is the image of its truth function under some atomic valuation. If the image is T [F], the statement is *valid* [invalid]. If a statement is valid under [all/some] atomic valuations, the statement is [tautologous/satisfiable]. If a statement is not valid under any atomic valuation, its denial is a tautology. If all atomic valuations satisfying α also satisfy β , and vice versa, α and β are *tautologically equivalent*, denoted $\alpha \leftrightarrow \beta$.

Proof (adapted from Halmos and Givant 1998: §13). An *axiom* is a statement asserted true without proof. The rule of detachment is: If α and $\alpha \rightarrow \beta$ are both tautologies, then β is also a tautology. Let $i, j, k, n \in \mathbb{N}$. A *formal proof A (proof)* is an ordered sequence of statements with typical element α_k $1 \leq k \leq n < \infty$. For each α_k , $i, j < k$, α_k is either (i) a substitution instance of some definition, axiom, or already proved consequence, resulting from the application, often tacit, of R1 and R2 (§3.1), or (ii) the result of applying detachment to some pair α_i and α_j . (i) suffices for equational logics such as boundary logic, for which detachment is just a special case of *propositional consequence* (§5.3).

α_n is a *theorem* iff α_n is the last statement of some proof A . If a proof A exists for α , α is *provable*. CTV is *sound* (all provable formulae are valid), *complete* (all valid formulae are provable), and *decidable* (there exist algorithms, e.g. TVA, for determining the satisfiability of any finite formula.) The *primitive basis* (DeLong 1971: 91) of a logic consists of primitive symbols, defined constants, rules of formula formation, axioms and rules of inference, and a truth definition. A *model* is an interpretation of a logic under which all its formulae come out true; see Suppes (1957: §4.2), Mendelson (1997: §2.2).

Calculus of Quantified Individuals, CQI. An *variable* stands for any member of a nonempty collection (*domain [of interpretation]*) of physical or abstract *individuals*, each having a *name*. A *term* is a name, variable, or a function thereof. Lower case letters denote terms. *Predicate letters* are upper case. Associated with each predicate and function letter is a nonnegative integer called its *arity*. A predicate [function] letter with an arity of 0 is a statement variable [*constant*]. An *atomic formula* (aka predicate) consists of a predicate letter followed by *arity* terms. An atomic formula with [more than] one term is [*polyadic*] *monadic*. \forall [\exists] is the *universal [existential] quantifier*. A quantifier operates on the variable that immediately follows it; $\forall x$ [$\exists x$] translates as “for all [for some] x ”. An atomic formula has a truth value only if its variables are quantified. A CQI formula consists of quantifiers, and atomic formulae linked by truth functors.

Let α, β be arbitrary FOL formulae. $\exists x \alpha \stackrel{\text{df}}{=} \neg \forall x [\neg \alpha]$, so that there is in fact only one quantifier. Let x and y be vectors of variables, of unspecified dimension. Let Q_i be one of \forall or \exists , and let $Q(x)$ be a string of the form $Q_1 x_1 Q_2 x_2 \dots$ known as a *prefix*. Let a *matrix* $M(x, y)$ consist of atomic formulae and truth functors but no quantifiers, with each x_i and y_i appearing at least once. Q_i *binds* x_i , and x_i is a *bound* variable; the y_i are *free*. If y has dimension 0 (≥ 0), then α is *closed* (*open*). CTV formulae are the special case when both x and y have dimension 0. The *prenex* form of α is then $Q(x)M(x, y)$, The *scope* of Qx_i is $M(x)$ by default, or overridden by parentheses. If x has dimension 1, and if α is closed and does not lie within the scope of another quantifier, then α is an *elementary quantification*. Writing $\forall y_1 \forall y_2 \dots$ to the left of an open formula results in its *universal closure*. α is *valid* (is a “law of logic”) if it evaluates to T for all values of all nonempty domains.

CQI requires a mere two axioms in addition to those for the CTV: UI, $\forall x \alpha(x) \rightarrow \alpha(a/x)$, and EI, $\exists x \alpha(x) \rightarrow \alpha(a/x)$. In the case of EI, a must be new to the proof at hand. FOL is provably sound (Hunter 1971: §42), complete (§46), and undecidable. Some necessary conditions for a FOL formula to be undecidable include a domain with infinitely many individuals, a prefix with at least one \exists preceding one or more \forall , and a matrix that is not a substitution instance of a monadic formula. The necessary and sufficient conditions for decidability are not known. FOL *with identity* includes a primitive dyadic predicate, denoted by infix ‘=’, which is axiomatically reflexive ($x=x$) and Euclidian $[(x=z) \wedge (y=z)] \rightarrow (x=y)$.

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