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FOUNDATIONS OF MATHEMATICS

ABSTRACTIONS POSTULATE

In "STRUCTURES OF MIND" we have defined the Abstractions Postulate (AP) as condition of meaningfulness of abstract constructs:

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Abstract, symbolic constructs may be justified solely by their capacity to coordinate events which represent their unique meaning and justification, where coordination of events implies considering them in their context, i.e. upon their background of continuum.

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Mathematical theories are abstract structures and, as such, may get their meaning and justification only by being founded in AP. Yet, this criterion, although necessary, is not sufficient to distinguish mathematics from other rational abstract structures also founded in AP. Therefore we shall start by trying to define mathematics before discussing

- foundation of mathematics in continuum,
- falsifiability of mathematical axioms,

preceded by a brief revue of the concurrent foundational crisis.

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MATHEMATICS

Precise lexicographic definition of "mathematics" is unattainable for the following reasons:

By virtue of AP, words and other symbolic linguistic constructs are not known or meaningful by themselves, but point to their underlying events which embody their meaning.

Dictionaries try to define them either with help of presumably better known synonyms, or intensionally - by giving a superclass and specific characteristic, defining for instance a bicycle as a two-wheeled velocipede.

Now, mathematics does not have a better known synonym and is too general and too vast to admit a precise superclass and specific characteristic.

As all generalities it admits only a vague extensional definition by enumerating some of its typical subclasses and some characteristics shared by them. Thus, mathematics is, or consists of, geometry, arithmetic, algebra, algebraized geometry, calculus, vector and tensor calculi, topology, etc., dealing with eventtual qualities of shape, quantity and order, founded in the Abstractions Postulate and axiomatic in the sense of "DOGMATIC THEORIES AND AXIOMATIC MODELS" based upon the ERN Logic.

One may perhaps imagine arbitrary, unfounded chains of symbols and call it "mathematics", but we dismiss it and reserve the term "mathematics" for abstract structures compatible with the above extensional definition.

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Review of the foundational crisis

The crisis of foundations of mathematics is universally recognized and seen as the conflict of three mainstream views:

Goedel's pseudo-platonic reifications,
Brouwer's pseudo-intuitionism and
Hilbert's games with arbitrary chains
of characters.

However, they all share the foundations in discreteness embodied in the set theory. Fraenkel honestly laid the crisis at the door of the very concept of the set theory, rather than blaming internal quarrels among its shades.

As we have shown in "SET THEORY", the crisis manifests itself

-by innumerable versions and shades of the set theory, all equally inconclusive,

-by the failure to define the most fundamental concepts of the set theory, such as "set" and "number", in spite of over 100 years of efforts of the logical establishment, including Cantor, Zermelo, Fraenkel, Russell, Frege, Quine, Church, Tarski, Goedel, Hilbert, Brouwer and innumerable others,

-by the incongruous concept of continuum and fallacious procedures supporting it. As continuum is a crucial issue of the present chapter, we shall review the way the set theory deals with it.

Set theories define "continuum" as the set of real numbers. It certainly has nothing to do with the intuitive continuum of time/awareness which we posited in "TIME, AWARENESS AND EVENTS" as the ultimate foundation of the human universe,

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nor with the infinite continuous space of physical reality which we defined in "NATURAL MODEL". In order to distinguish the set theoretical numeric gimmick from the ontological foundation of human universe and physical reality, we shall put the "continuum" of real numbers in quotes.

Let's note, by the way, that "set" and "number" staying as yet undefined, the definition of "continuum" as the "set of (real) numbers" does not have a leg to stand upon.

The "continuum" is supported by the famous "Continuum Hypothesis": Calling A_0 and A_1 respectively the transfinite cardinals of sets of natural and real numbers, Cantor proved that A_1 is greater than A_0 and postulated the "Continuum Hypothesis" stating that there is no set whose cardinality falls strictly between A_0 and A_1 .

Poincare considered transfinity as a disease and Kronecker as scientific charlatanry. We cannot but agree with them and see the "Continuum Hypothesis" together with its underlying transfinity as delusional humbug.

Ontological foundations

In the next section we postulate that mathematics is founded in continuum which is the cornerstone of Second Enlightenment's ontology - the Relativistic Dialectic (RD). It seems therefore advisable to justify RD as the pertinent foundation of scientific models.

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Most ontologies are dogmatic, consisting of whimsical speculations a priori, aspiring to absolute truth, and high-handedly snubbing science, know-how and, above all, facts. "If the facts disagree with me then so much worse for the facts." - this Hegel's declaration may serve as motto of Dogmatism.

Yet, there exist rational ontologies, endeavoring to found their contemporary science and know-how and considering their presumptions as axioms verifiable or falsifiable by facts.

Kant's ontology springs to the mind. It has been briefly introduced in the "PREFACE". Here we wish to consider one of its axioms derived from the science of the First Enlightenment culminating in Newton's model:

Science was concurrently considered as absolute and certain, thus necessary. On the other hand, science is constructed by "synthetic" statements and only a priori statements may be necessary. Consequently, Kant postulated the axiom of existence of "synthetic" statements a priori. It persisted till the Second Enlightenment which falsified it by revealing the fuzzy and uncertain fabric of science.

Consequently, the particular assumptions of Kant's view became for us obsolete and falsified, but his method of deriving an axiomatic ontology from the current know-how stays a topical and well-advised example.

Einstein followed it conceiving his "Physical Reality", which is the kernel of the Relativistic Dialectic (RD) put forth in the present essay.

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It has still deeper and more intimate relations with the cutting edge of concurrent physics than those of Kant with Newton. Kant conceived a fair and pertinent ontology, which generalized and founded a posteriori the Newton's model. Einstein's "Physical Reality" for the first time in the history precedes and concretely underlies physics, which uses directly some ontological assertions as axioms fit for rigorous processing. Such is, for instance, the case of the ontological Covering Principle ("see NATURAL MODEL"), underlying the derivation of the General Relativity by means of the mental experiment of "Rotating Disk". The Extended Relativity would be unthinkable without being concretely founded in the ontology of the "Physical Reality" and in particular in its Covering Principle.

Arithmetics founded in continuous geometry

In "NATURAL MODEL" we have asserted:

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mind's faculty of putting every body situated in any arbitrary manner into contact with the quasi rigid continuation of a chosen body of relation B_0 is the basis of our intuition of space. In pre-scientific thinking, the solid earth's crust plays the role of B_0 . The very name geometry indicates that the idea of space is mentally connected with the earth considered as the relation body.

We shall find the intuition of space founded by rigid bodies at the base of Einstein's Covering Principle which requires physical distance to be measured, also in mental experiments,

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with physical rods complying with physical rules, such as the Lorentz Contraction. The Covering Principle underlies directly or indirectly the entire Extended Relativity.

Continuous space, its primacy and the discrete covering measurement rods are aspects of the fundamental Polarity Continuum/Discreteness (CD) defined in "TIME, AWARENESS AND EVENTS" as the elementary structure of human Universe.

In mathematical terms geometry symbolizes the basic intuitive image of continuous space and arithmetic - its discrete covering measurements. The ultimate foundation of mathematics is the continuum. Discrete concepts starting with that of "number" are founded in continuum and symbolize its covering measurements. Shortly, arithmetic is founded in geometry.

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If mathematics were restricted to arithmetic, we could stop here. But it extends beyond arithmetic over innumerable interrelated theories. In order to exemplify their ultimate foundation in continuum we shall recall the concept of foundation hierarchies.

Foundation hierarchies

Besides being founded in their own, "local" axioms, models may be founded in other "founding" models. By definition, a model or a discipline is "founded" in a "founding" one, when it accepts the latter's axioms and theorems as its own axioms.

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Thus, physics is founded in mathematics and does not derive the principles of calculus, of vectors, tensors, etc. but considers their mathematical formulations as axioms of its own models.

This foundation hierarchies stems from the ontological intuition of continuum, the primary aspect of the Polarity Continuum/Discreteness (CD), the fundamental construct of the physical and human reality ("NATURAL MODEL"). Hence, rational axiomatic models are ultimately founded in continuum.

In "SET THEORY" we saw the fallacies resulting from attempting to found mathematics in discreteness. In Tome 2 "FOUNDATIONS OF QUANTUM PHYSICS" we shall discuss the controversy between quantizing the fundamental continuum of field and attempts to found quantum physics in sheer discreteness without considering SPACE or field continuum.

Briefly, the Relativistic Dialectic posits rational models as ultimately founded in the intuitive continuum. Abstract structures lacking this foundation fall in the domain of irrationality, beyond the human universe of discourse.

In the following section we shall illustrate the reduction to continuous foundation with help of Euler's Formula, sufficiently complex and high up in the foundation hierarchy to be considered as typical.

We don't pretend that whole established mathematics is rational in the above sense.

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On the contrary, we see a lot of theories or theorems as irrational and meaningless plays with symbols, e.g. Banach-Tarski Paradox or Goedel's Theorem.

The following example may help to decide if some established theory can be considered as a rational model, or disregarded.

Euler's Formula

$$e^{ix} = \cos(x) + i\sin(x)$$

Feynman called Euler's formula "our jewel" and "one of the most remarkable, almost astounding, formulas in all of mathematics". Indeed, on the one hand it relates completely dissimilar concepts of exponential and trigonometric functions with help, moreover, of apparently out-of-the-way imaginary numbers. On the other hand, in spite of its concise form it involves astounding variety and extent of mathematical domains, to mention complex numbers, exponential function, trigonometry hyperbolic and circular, algebra, algebraic (analytic) geometry, calculus with differentiation, integration and infinite series, etc.

Sharing Feynman's admiration, we note that the ramifications of Euler's Formula make it particularly suitable to exemplify our postulate of the ultimate founding of rational models in the ontological continuum. The demonstration is surprisingly simple:

1. The proof of Euler's Formula consists in the infinite series of both sides being equal.
2. Infinite series are founded in calculus.

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3. Calculus is the symbolic, abstract map of the fundamental structure of human reality - the CD (Continuous/Discrete) polarity - differentiation representing continuum and integration - its discretization.

Besides the Euler's Formula, there are innumerable rational mathematical constructs clearly regressing to continuum. Let's mention the Gauss-Ostrogradsky theorem, which allows to express several "continuity equations" in alternative differential or integral forms.

Falsifiability of mathematical axioms

Having justified the postulate of rational mathematical models being founded in continuum, we are still facing the open problem of falsifiability of mathematical axioms. While axioms of natural sciences and applied mathematics are falsifiable by physical facts, one often objects that pure mathematics are entirely mental and don't deal with any facts suitable to falsify them.

Our ontology rebuts this objection with two arguments postulated in "NATURAL MODEL" in accord with Einstein's "PHYSICS AND REALITY":

1. In spite of the illusion of "objective reality" human Physical Reality is entirely immanent and mental, which puts it at parity with mathematics.
2. ...imagery supports "mental experiments", whose events are projections of abstract concepts, of emotional impressions or images brought about recursively by analogy with known ones.

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Mental experiments support most, if not all human creativity. As it may appear rather complex, we shall illustrate it with the example of hyperbolic geometry.

During 2000 years Greek, Persian, Arab and modern European mathematicians tried to derive the Parallel Postulate from other Euclidean axioms. All those trials failed, but several, especially those proceeding by reductio ad absurdum, came up with strange byproducts, which finally added up to the complete and consistent Hyperbolic Geometry published by Lobachevsky and Bolyai. It replaces the Euclidean Postulate of a single parallel with any number of distinct parallels higher than one. One of its theorems states that the ratio of circumference to diameter of a disk is greater than π .

In concert with covering principle and Lorentz Transformation, the mental experiment of rotating disk results in this ratio greater than π , thus postulating the hyperbolic SPACE of spinning cosmic referentials and creating the General Relativity. Further development generalized the hyperbolic SPACE to Riemann SPACE defining pointwise flat, elliptic and hyperbolic geometries, representing centrifugal and centripetal areas of Cosmos as well as their transitions.

So far it was a pure mathematical mental experiment including the purely abstract concept of Cosmos. Physical factual verifications such as Einstein's lens came years later emphasizing the ontological equivalence of mathematical and physical mental experiments.

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Postface

We believe to have shown the equivalence of foundations of mathematics and physics in ontological continuum. In other terms, mathematics does not have, nor need particular foundations, different from the general foundations of rational symbolic constructs.

History seems to confirm our view. Euclides, Pythagoras, Muhammad ibn Musa al-Khwarizmi, Descartes, Newton, Leibnitz, Euler, Gauss, Riemann, Dirac did their mathematics without needing any special foundations, did not wait for the so called analytic philosophers to create them, nor noticed them, if contemporary.

Another confirmation of our view may be found in the ultimate Frege's disappointment. In 1923 he came to the conclusion that the aim he had set himself throughout most of his career, namely to found arithmetic in (predicate) logic, was wrong. He decided instead (like ourselves) that one had to base the whole of mathematics on geometry. He began to work on these ideas but had not progressed far by the time of his death.