I-C3-1

SET THEORY

DEFINITIONS

Fraenkel defines Set Theory as "applied first order Predicate Calculus" enlarged with the one specific primitive symbol, the diadic "membership" predicate/operator "$\in$". We shall read $x \in y$ as "x is member of (set)y" or "(set)y contains (member)x". Set Theory being thus founded in Predicate Logic we could - in the light of the previous chapter - dismiss it straight off as another noumenal phantasm.

One cannot, however, exclude a priori the possibility of some out-of-the-way peculiarities of the membership operator doctoring the Set Theory and somehow making it rational. We shall, therefore, examine the implications of the membership operator, starting by asking what are the involved "set" and "member" concepts.

Strangely enough, everybody seems to steer clear of defining "set". Fraenkel and Bar-Hillel deal mainly with antinomies and with innumerable axiomatic systems trying to avoid them in different ways, but don't define "set". The closest to a definition is Levy's following non-definition:

By set we mean a completely structure-free set, and therefore a set is determined solely by its members

We shall not dwell on this unfortunate formulation looking like a typo in otherwise well and clearly written book, but the fact remains - we still don't know what is "set, what is "member", nor what it means "to have members".

The way out of the deadlock consists, as usually, in replacing the definition with a symbolically expressed axiom. And so we arrive at the first axiom of the set theory:

Axiom of Existentiality (Frege 1893)

$\forall x(x\in y\leftrightarrow x\in z)\rightarrow y=z$

(if y and z have the same members they are equal).

Next Levy's quote:

The existence of sets: Now we face the question of finding or constructing sets. We want any collection whatsoever of objects, i.e. sets, to be a set. This is not a precise idea and therefore we cannot translate it into our language. We must therefore be satisfied with a somewhat weaker stipulation. We shall require that every collection of sets "specifiable" in our language is a set; i.e., for every statement of our language this collection of all objects which satisfy it is a set. We shall by no means assume that it is necessarily true that all sets are specifiable; moreover by introducing the Axiom of Choice we shall require the existence of sets which are not necessarily specifiable. The requirement that all specifiable collections are indeed sets is the following one.

Axiom of Comprehension (Frege 1893)

$\exists y\forall x(x\in y\leftrightarrow F(x))$

(x is member of y IFF it satisfies the formula F(x))

No limitation specified on the formula F(x), an instance of the axiom may be chosen taking

$F(x)=x\notin x$
(x is not member of itself), leading to the contradiction known as the Russell's Paradox:

Theorem: Russell's Paradox
$\neg\exists y\forall x(x\in y\longleftrightarrow x\notin x)$

It's a theorem of predicate logic, since we don't use in the proof any axioms of set theory.

Proof:
Suppose y is a set such that
$\forall x(x\in y\longleftrightarrow x\notin x)$
Then, since what holds for every x, holds in particular for y we have:

$y\in y\longleftrightarrow y\notin y$

which is a contradiction
(y is member of itself IFF y is not member of itself).

Let's resume: The very beginning of the set theory is completely muddled. "Set" is "defined" as "structure-free set" determined by its "members(?)". "We want any collection whatsoever of objects, i.e. sets, to be a set". What are "objects", what are their "collections"? Any "collection(?)" of "sets(?)" is a "set(?)", but what, for heaven's sake, is "set"? One takes with a grain of salt the requirement of "sets" being "specifiable" in "our language" (Predicate Logic plus membership operator).
The foundation of mathematics and science depending on possible arbitrary changes of "our language" sounds rather thin.

And the axiom implementing this requirement leads right away to an antinomy proving that something is rotten at the very base of the Set Theory and of its founding Predicate Logic.

From this initial moment the Set Theory becomes rat hunting for antinomies, just as its base, the Predicate Logic became rat hunting for "reflexive fallacies". Fraenkel, prudently dodging the definitions starts his elucidation of the Set Theory by detecting and hunting rats such as antinomies of: Russell, Cantor, Burali-Forti, Richard, Grelling and, of course, the Liar, prudently dressed up in new but equally fictitious clothes, leaving him naked as before.

FAKED AXIOMS

Antinomies are hunted by means of dogmatic decrees usurping the name "axioms" in spite of lacking the principal feature thereof, to wit the falsifiability. They simply embody the Wishful Thinking Principle: -When X disturbs us, we issue an "axiom" boiling down to "X is verboten"- usually dressed up in impressing verbiage and followed by complex circular procedure called "proof".

For instance, Zermelo disposes of the Russell paradox by means of the "Axiom of Separation" and a theorem:

Axiom of Separation:"Whenever the propositional function -(x) is definite for all elements of a set M, M possesses a subset M' containing as elements precisely those elements x of M for which -(x) is true".

I-C3-5

The Theorem: "Every set M possesses at least one subset M0 that is not an element of M". Let M0 be the subset of M for which, by Axiom of Separation, is separated out by the notion "x is not member of x". Then M0 cannot be in M:

Now, that clearly boils down to the Wishful Thinking decree: "Russell paradox is verboten". The following tedious "proof" is a superfluous, circular tautology:

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Proof

M0 cannot be in M:

1. If M0 is in M0, then M0 contains an element x for which x is in x (i.e. M0 itself), which would contradict the definition of M0.
2. If M0 is not in M0, M0 is an element of M that satisfies the definition "x is not member of x", and so is in M0, which is a contradiction.

So M0 cannot be in M, hence not all objects of the universal domain B can be elements of one and the same set. "This disposes of the Russell antinomy as far as we are concerned".

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However, not all of us seem convinced. One tends to find the Wishful Thinking procedure void of sense and "the domain B" coming out of the blue mysterious and unrelated to the issue.

One may therefore prefer the bold and simple procedure inspired by cutting the Gordian knot and consider Russell's paradox as the proof that Russell's collection is not a set!

Using this approach we must legalize some collections of sets that are not sets, or, as says Levy, some collections of objects not being objects themselves. In order to denote with a general term all set- and non-set collections one introduces the term "Class", postulating that all specifiable collections are classes, which may or may not be sets.

We still don't know what are "collections" and "objects" and get an additional puzzle - which decree commands the "collections of objects" to be or not to be "objects" themselves.

Most axioms added hereafter to the ST seem to embody the Wishful Thinking Principle and to fall into three principal types:

1.Axioms banning culprits of antinomies from the community of sets, e.g. the dealing with Russell's antinomy described above.

2.Hijacking of mathematical constructs and principles and presenting them post factum as founded in the Set Theory. We may mention here neighborhood and limit, which worked rigorously and efficiently for years, but suddenly got contested unless founded in the axiom of choice.

3.Dealing with infinity and continuity. It intersects with the preceding type, because most non-trivial mathematical problems deal with infinity and continuum. However, there is a particular set-theoretical problem of constructing continuum from discrete collections, which deals with transfinite numbers defined by Cantor as:

cardinal numbers or ordinal numbers that are larger than all finite numbers, yet not necessarily absolutely infinite.

Calling A0 and A1 respectively the transfinite cardinals of sets of natural and real numbers, Cantor proved that A1 is greater than A0 and enonced the Continuum Hypothesis stating that there is no intermediate cardinal number between A0 and A1. Wishful Thinking elevated the hypothesis to an axiom and transfinity was saved, having nevertheless induced shock waves of unusual magnitude.

Poincare condemned the theory of transfinite numbers as a "disease" from which he was certain mathematics would someday be cured and Kronecker even attacked Cantor personally, calling him a "scientific charlatan", a "renegade" and a "corrupter of youth".

One cannot help feeling uneasy in face of transfinity and of the awkward dealing with "collections" in the stage of opening definitions of the set theory. It seems to be due to the current foundational crisis of mathematics, the third one, according to Abraham A. Fraenkel and Yehoshua Bar-Hillel, as quoted in the following paragraph.

I-C3-8

THE THREE CRISES

The twentieth century is not the first period in which mathematics underwent a foundational crisis. It might add to the perspective in which contemporary antinomies should be looked upon if prior crises are, if only briefly, sketched.

In the fifth century B.C., only a short time after mankind obtained one of the most brilliant achievements in its history, viz. the development of geometry as a rigorous deductive science, two discoveries were made that were extremely paradoxical: The first was that not all geometrical entities of the same kind are commensurable with each other, so that, for instance, the diagonal of a given square could not be measured by an aliquot part of its side (in modern terms, that the square root of 2 is not a rational number); the other were the paradoxes of the Eleatic school (Zenon and his circle) developing with many variations the theme of the non-constructibility of finite magnitudes out of infinitly small parts.

This crisis shocked the Greek mathematicians into obtaining two more brilliant achievements: the Theory of Proportions, as contained in books 5 and 10 of Euclid's Elements, and the Method of Exhaustion, as invented by Archimedes, that was nothing less than a strict, though not sufficiently general forerunner of modern theories of integration.

Their Theory of Proportions should have enabled the Greeks to define irrational number and develop, accordingly, an arithmetical theory of the Continuum; somehow they did not quite make it.

The Greek Theory of Proportions was soon forgotten - so much so that when rigorous arithmetical theories of irrational numbers were constructed in the second half of the 19th century, one was not at first aware of the fact that these methods were not in principle much different from those already in the possession of the Greek mathematicians two thousand years earlier. Before that, in the 17th and 18th centuries, the great power and fruitfulness of the newly invented calculus led most mathematicians of those times into feverish applications of the new ideas without caring much for the solidity of the basis upon which the calculus was founded. However, the shakiness of this basis became clear at the beginning of the 19th century, constituting the Second Crisis in the foundations of mathematics.

In order to overcome this crisis, Cauchy, in the eighteen thirties showed how to replace the irresponsible use of infinitesimals by a careful use of limit, while Weierstrauss and others, in the sixties and seventies, demonstrated how all of analysis and function theory could be "arithmetized". This solidification of the foundations was so successful that Poincare, in an address delivered in 1900 before the Second International Congress of Mathematicians on the rôle of intuition and logic in mathematics, could proudly claim that mathematics had by then acquired a completely solid and sound basis. In his own words: "Today there remain in analysis only integers... Mathematics... has been arithmetized... We may say today that absolute rigor has been obtained."

Ironically enough, at the very same time that Poincare made his proud claim, it had already turned out that the theory of the "infinite system of integers" - nothing else but a part of the set theory - was very far from having obtained absolute security of foundations.

More than the mere appearance of antinomies at the basis of the set theory, and thereby of analysis, it's the fact that the various attempts to overcome these antinomies...revealed a... surprising divergence of opinions and conceptions on the most fundamental mathematical notions, such as set and number themselves, which induces us to speak of the Third Fundational Crisis that mathematics is still undergoing.

CONTINUITY AND DISCRETENESS

The Third Fundational Crisis brought up by Fraenkel and Bar-Hillel is also known as the Fin-de-Siecle Crisis.

19th Century Physics and Logic were dominated by reaction of Dogmatism against Rationality of the First Enlightenment. Physics was founded in the dogmatic delusion of Aether. Logical systems culminated in Predicate Logic (PL).

This noumenal, ill founded Logic was assumed to be the universal, absolute foundation of transcendental "Reality" and of human knowledge thereof. All areas of Science were assumed to be recursively founded in PL, although Mathematics stayed for some time unfounded.

Finally, in 1873 Cantor conceived the Set Theory which supplied the missing link andcompleted the "Cantorian Paradise", which represented the edifice of Science solid and stable as Cheops pyramid, bestowing upon scientists the gratifying conviction that Logic, THEIR Logic, explains and determines the "Real Universe".

At the end of the 19th century two breaches appeared in the pyramid setting off the Fin-de-Siecle Crisis of Thought which triggered the Second Scientific Revolution:

1. In Physics the Michelson-Morley experiment showing invariance of the speed of light and thus falsifying the mechanistic Dogma.
2. Russell's Paradoxes calling in question the foundations and sense of PL and founded in it Set Theory.

Question arose, if apparently local breaches in otherwise solid pyramid may be patched locally, or should lead to global reconsideration of the whole structure.

In Physics Einstein has chosen the global approach, revised entire Physics and laid the cornerstone of the Second Enlightenment with its new Rationality. Without generating any "surprising divergences of opinions and conceptions", the new physics encompasses four or five principal models, complementary, giving uniform satisfaction and sharing the same foundations.

I-C3-12

Logicians did not notice that Russell's Paradoxes did not wreck Logic as such, but only the PL and the set theory, and try in vain, till our days, to patch them rather than to scrap the wrecks and to conceive in their place new rational foundations.

As to "divergences of opinions and conceptions", we may count over 40 TYPES of theories, formalized, intuitive, logistic, (not)-axiomatizable, (un)-decidable, elementary, first- second- ... nth-order, formally (in)-consistent, (in)-complete, (in)-completable, recursively incompletable, semantically (in)-consistent, semantically (in)-complete, (non)-categorical, relatively categorical, etc. Each type encompassing several theories, a lifetime would not suffice to learn them all, due on the one hand to their obscurity and, on the other hand, to new ones popping up while you struggle with one. And after 100 years of this proliferation none can tell us - to believe Fraenkel and Bar-Hillel - what's "set" and what's "number".

In face of this unbridled chaos one is tempted to look at the stable and robust foundations of physics and to see if and how they could assist those - hopelessly confused - of mathematics.

Our inquiry of time and events (Part "A_FOUNDATIONS") brought to light the CD (Continuum/Discreteness) Polarity as the basic element of the human universe of discourse and the foundational primacy of its continuous term. Following this primacy, physics is founded in the continuum of SpaceTime-Field, discretizing it to singularity areas and particles.

I-C3-13

Set theories inverse this natural primacy and are ill founded in discrete "collections" ascending to continuum via phantasmic transfinities. As long as this primacy violation persists, mathematics will lack proper foundations. In other terms arithmetics should be founded in geometry and not the other way round as the establishment pretends.

Another confirmation of our view may be found in the ultimate Frege's disappointment. In 1923 he came to the conclusion that the aim he had set himself throughout most of his career, namely to found arithmetic in (predicate) logic, was wrong. He decided instead (like ourselves) that one had to base the whole of mathematics on geometry. He began to work on these ideas but had not progressed far by the time of his death

The proper foundations of mathematics will have to be conceived from the scratch, based upon continuum, locally discretized whenever necessary.

References:

1.Abraham A. Fraenkel and Yehoshua Bar-Hillel "Foundations of Set Theory".
2.Azriel Levy "Basic Set Theory".