The Hunting of Another Snark

$a^n + b^n = c^n$

Tony Thomas

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No one knows whether Pierre de Fermat really did have a proof of his famous conjecture that would justify it being called a theorem. However, he declared that he did in his copy of Diophantus' Arithmetica. The original copy was lost but the marginalia was reported by his son to be as follows.

It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second into two like powers. I have discovered a truly marvellous proof of this, which this margin is too narrow to contain.

It would be churlish to suppose that Fermat had no genuine proof, but it is strange that no record of such a proof exists other than his provocative marginal note. Over 350 years elapsed before Andrew Wiles produced his complex proof of Fermat's conjecture, and this was only achieved after years of research into the problem. Wiles's proof is difficult and draws upon several esoteric branches of mathematics. This renders it incomprehensible to all but the highly trained mathematician. Unfortunately I am not among these.

Fermat's conjecture appears to be very simple, and states that the sum of two integral numbers raised to a third or higher power can never equal an integer raised to the same power ie $a^n + b^n \neq c^n$ when $n > 2$. This qualification is necessary because there are infinitely many solution to the equation when $n = 2$. In other words there are innumerable examples of Pythagorean triangles with three integral sides. The simplest of these is the (3,4,5) triangle, which was known to the Babylonians.

Fermat's conjecture applies to all dimensions higher than two and it is disappointing that no examples of $a^n + b^n = c^n$ have ever been discovered for dimensions higher than two. A cursory examination of the problem soon convinces the enquirer that it is unlikely that there could be any solutions, but proving that this is so requires more than familiarity with the problem and mere intuition.

The conjecture poses a unique problem because it seeks to prove a negative: that there are no solutions to the equation. Searching for a needle in a haystack is difficult enough but becomes nigh on impossible when the haystack is infinite. However, proving there is no needle raises the problem to a still higher level.

In his poem, *The Hunting of the Snark*, Lewis Carrol seems to wrestle with a similar problem. A group of bumbling adventurers sets out in a boat (with the bowsprit on the stern) in search of a mythical but deadly creature: the Snark. The journey is fraught with dangers, including encounters with the Jubjub, the Bandersnatch and possibly the Boojum. Such fabulous creatures are reminiscent of the strange mathematical forms that arise in pursuit of the theorem that would prove Fermat's conjecture true.

Among these hazards are fallacies, facile assumptions, misconceptions, transcription errors and the confusion of one symbol with another, not to speak of downright blunders of the crudest kind. The imagination and creative intuition flies forward at lightning speed, leaving a trail of errors in its wake. Triumphant sightings of the Snark inevitably turn out to be the Boojum. Lewis Carrol might well have named some of these creatures as ‘fallunders’, blundacies’ or ‘crudumptions’ according to his theory of portmanteau words with joint meanings.

Fermat’s conjecture poses an intractable problem, like a smooth cliff face that offers the climber no safe handhold. Expanding the original form into a more elaborate one provides plenty of scope for beginning the climb towards the inaccessible peak of the solution. The binomial theorem is an obvious route but one that soon leads into a wilderness of ever increasing complexity.

For the beginner, a useful preliminary is to study the Pythagorean forms in detail ie the solutions to $a^2 + b^2 = c^2$. The basic (3,4,5) triangle is not alone: for every odd number there is a solution that is easy to calculate. Examples of these ‘primitive’ forms are (3,4,5), (5,12,13), (7,24,25), (9,40,41) etc. Each of these forms can be multiplied without limit to provide an infinite number of solutions eg (6,8,10), (9,12,15) etc. This fundamental principle means that the existence of a single primitive solution for $n > 2$ implies the existence of infinitely many solutions. The failure to detect any solutions strongly suggests that the Snark is well hidden in its infinite maze.

This process of enquiry reveals the principles that would have to apply to higher dimensions. For example, the primitive forms must be (odd,even,odd) and the only possible alternative is (even,even,even). This simple notion of parity consistency (odd + odd = even) and (even + even = even) leads on to the more complex idea of how numbers combine according to the residues generated by moduli higher than 2. Choosing a line of enquiry to pursue holds out the prospect of a wrong direction that could involve years of wasted time or a quick route to a solution.

The brute force of the computer provides empirical evidence that $a^n + b^n \neq c^n$, although many of the results are tantalisingly close. However, the computer’s power is soon overcome as its capacity to deal with large numbers becomes exhausted. The refinements of logic and mathematics cannot be replaced by technology, but the computer is useful for presenting patterns to the eye that may not be evident to the unaided intellect.

Adopting many lines of enquiry and gathering insights along the way is an invaluable process. Einstein’s aphorism, “imagination is more important than knowledge” lies at the heart of the creative process although little progress can be made without a sufficient understanding of mathematics and logic. Devising solutions to Fermat’s conjecture has long been the hobby of cranks and incompetent mathematical amateurs. The number of erroneous solutions has multiplied exponentially over the centuries and contributions by eccentric geniuses continue to plague mathematics departments around the world.

This poses a dilemma for the dedicated hunter of the Snark. Should the enquirer who has survived the clutches of the Bandersnatch and the Jubjub bird continue with the search, despite the continual transformation of the elusive Snark into the ubiquitous Boojum or give up the hunt knowing that no one would believe that another Snark had been captured and rendered harmless. Even having the Snark in one's clutches would expose the hunter to the ridicule of real mathematicians who know that the only one in existence has already been captured by Andrew Wiles, and lies caged somewhere in Princeton University. If it walks like a Snark, and quarks like a Snark it may well be a Snark, so without further ado I offer the Boojum below for your tender consideration.

Fermat's Last 'Theorem'

It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second into two like powers. I have discovered a truly marvellous proof of this, which this margin is too narrow to contain.
Pierre de Fermat

The conjecture is:

$$a^n + b^n \neq c^n \qquad (a,b,c,n \in N), (a \neq b \neq c), (n > 2)$$

In seeking to prove this conjecture, which can be denoted by FLT, the contrary assumption ~FLT is made in the hope of refuting it and establishing FLT *reductio ad absurdum*.

Restating the conjecture in contrary form:

$$\sim\text{FLT: } a^n + b^n = c^n \qquad (a,b,c,n \in N), (a \neq b \neq c), (n > 2)$$

When $(a,b,c,n \in N)$ and $(n > 2)$ there exist values of a,b and c which satisfy $a^n + b^n = c^n$

Rearranging the terms:

$$c^n - b^n = a^n$$

Let $c = b + k$ $\qquad (k \in N)$

The finite difference $c^n - b^n$ then becomes $\Delta_k b^n = (b + k)^n - b^n$ so that

$$\sim\text{FLT} \Rightarrow a^n = \Delta_k b^n = (b + k)^n - b^n$$

If it can be shown that $a^n \neq \Delta_k b^n$ then FLT is proven. The focus of the proof, therefore, is directed towards this end. But first some auxiliary theorems are required.

Theorem 1

$a^n \neq b^{n-1}$ (a,b,n $\in$ N) (n > 2)

Suppose there are integral solutions to $a^n = b^{n-1}$ then $a = b^{(n-1)/n}$
Now, if b has an integral nth root β then $a = \beta^{n-1}$. But in this case $\beta = a^{1/(n-1)}$ and $\beta^n = a^{n/(n-1)}$ so a must have an integral (n-1)th root α such that $\alpha^n = \beta^n$ in which case $\alpha = \beta$ and a = b. But the hypothesis is that $a^n = b^{n-1}$ so $a^n = a^{n-1}$ which is impossible, so the theorem is proved *reductio ad absurdum*.

Theorem 2

$a^n \neq b^p$ (a,b,n $\in$ N) (p $\notin$ N) (n > 2) (n-1 < p < n)

Suppose there are integral solution to $a^n = b^p$ then $a = b^{p/n}$.
Since (n-1 < p < n) p is indivisible by n and so a cannot be an integer. This contradicts the assumption a $\in$ N, so the theorem is proved *reductio ad absurdum*.

Proof of Fermat's conjecture

The primary goal is to show that $a^n \neq \Delta_k b^n$ (a,b,k,n $\in$ N)

$\Delta_k b^n$ is a function of the integral variable b and is itself an integer. The function constitutes the set of points (b, $\Delta_k b^n$). Each of these points is associated with a unique, continuous, exponential function $y = x^p$.

The exponent p is defined by the point (x,y) as p = logy/logx, so that (x,y) lies on x^p . If x and y are integers then the point (x,y) is unique in that no other point on x^p is an integral pair except (0,0) and (1,1).

Let $p_b = \log\Delta_k b^n/\log b$ so that the point (b, $\Delta_k b^n$) is the unique integral pair on the curve b^p greater than (1,1). Consequently, there is a unique integral solution to $\Delta_k b^n = b^p$ at the point (b, $\Delta_k b^n$).

Suppose $a^n = \Delta_k b^n$, then $a^n = b^p$. But Theorem 2 asserts that $a^n \neq b^p$ so $\Delta_k b^n \neq a^n$ *reductio ad absurdum*.

Now ~FLT $\Rightarrow a^n = \Delta_k b^n = (b + k)^n - b^n$

But $\Delta_k b^n \neq a^n$

Therefore ~FLT is contradicted and FLT proved *reductio ad absurdum*.

Theorem: $a^n + b^n \neq c^n$ (a,b,c,n $\in$ N), (a $\neq$ b$\neq$ c), (n > 2)

Notes on the proof

The key idea used in the proof is that there is an infinity of continuous functions $y = x^p$ between x^{n-1} and x^n. A subset of these functions can be defined to include all the integral pairs $(b, \Delta_k b^n)$, where each x^p contains a single point of $\Delta_k b^n$. Since this set accounts for every point of $\Delta_k b^n$ none of these points can lie within a^n because $a^n \neq b^p$. The chart below depicts the intersection of the constructed power series x^p and $\Delta_k b^n$. The power series can be thought of as construction lines in a geometrical proof like Pythagoras's theorem, although there are infinitely many of them in the FLT proof.

Theorem 1 does not apply to the infinity of possible solutions when $n = 2$ because $a^n = b^{n-1}$ reduces to $a^2 = b$. However, theorem 2 does apply but simply asserts that none of the exponential functions between a^1 and a^2 can contain integral members of a^2. The fact that the finite differences of a^2 are linear accounts for the existence of solutions for $n = 2$ and for the absence of solutions for higher dimensions.

The Fermat problem can be looked at as two separate but more general problems:

(1) $f(a) + f(b) = f(c)$ and (2) $f(a) = g(b)$

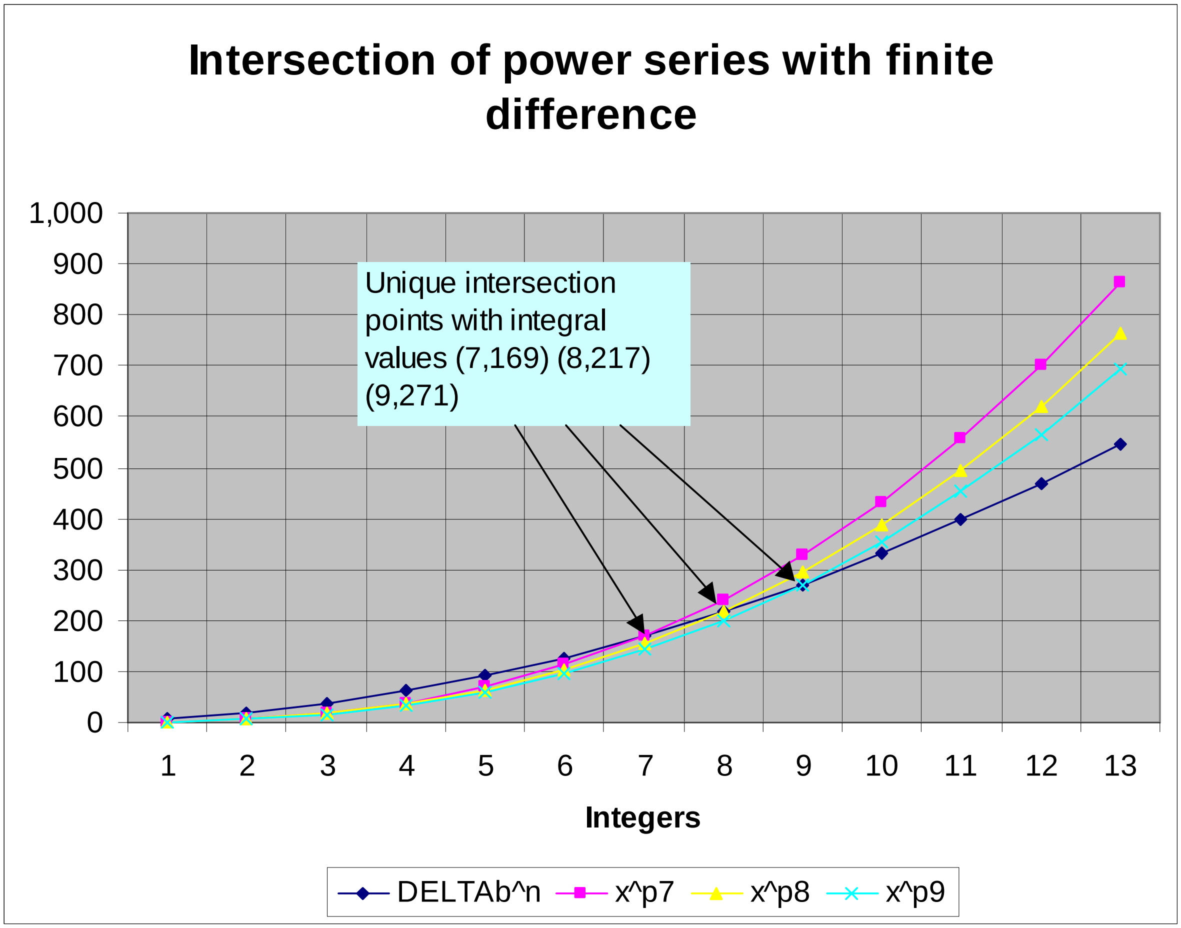
The first becomes the second on noticing that $f(c) – f(b) = f(a)$ and redefining $f(c)$ as $f(b+k)$.

The second case is much more general in asking whether two different functions can have integral solution, and if so according to what conditions. The answer depends on the way two different functions $f(a)$ and $g(b)$ can have integral solutions. The general answer to this wider question is given by the following theorem.

Theorem 3

If two different functions $f(x)$ and $g(x)$ intersect at a point $(a,f(x)) = (b,g(x))$ then there is an integral solution to the equation $f(x) = g(x)$. So $a = b$ and $f(a) = g(b)$ at each point of intersection and there are solutions defined by the points of intersection. Integral solutions are also possible when $f(x) = g(x)$ but $a \neq b$. In this case $x^{pi} = x^{pj}$.

The Fermat conjecture is a special case of $f(a) = g(b)$. Given any function $f(x)$ the internal division $f(a) + f(b) = f(c)$ can be made and the same question posed as in the Fermat conjecture. The solution can be found by the method applied to the Fermat problem ie $f(b+k) – f(b) = f(a)$. $f(b+k) – f(b)$ and $f(a)$ are treated as two distinct functions and each matched with a set of exponential functions x^{pi} and a set x^{pj} to show that there are no integral solutions other than the exceptions stated in theorem 3.



Intersection of power series with finite difference

Unique intersection points with integral values (7,169) (8,217) (9,271)