A DECIDABILITY THEOREM

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Abstract. The aim of this investigation is to show that supposed assumptions in the original proof of Gödel's First incompleteness theorem allow to infer a decidability of formulas that were asserted as undecidable in the theorem; moreover, we will prove that these decidable formulas are provable.

Gödel's First incompleteness theorem reads as follows [Gödel 1931b (engl.), p. 98]:

Theorem 1. *(Incompleteness theorem). For every ω-consistent recursive class $\varkappa$ of formulas there are recursive class sings r, such that neither υ Gen r nor Neg(υ Gen r) belongs to Flg($\varkappa$) (where υ is the free variable of r).*

Let us express the theorem by means of the logical symbolization:

$$\forall\varkappa\Big[recursive(\varkappa) \quad \& \quad {}_{\omega}consist(\varkappa) \quad \& \quad \exists r\Big(\big(r\ \varepsilon\ \varkappa \quad \& \quad recursive(r)\big) \supset \overline{\upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)} \quad \& \quad \overline{(Neg(\upsilon\ Gen\ r))\ \varepsilon\ Flg(\varkappa)}\Big)\Big].$$

Because r does not occur as free in $recursive(\varkappa)$ and ${}_{\omega}consist(\varkappa)$, given above formal expression of the theorem may be rewritten as:

$$\forall\varkappa,\ \exists r\Big[\big(recursive(\varkappa) \quad \& \quad {}_{\omega}consist(\varkappa) \quad \& \quad r\ \varepsilon\ \varkappa \quad \& \quad recursive(r)\big) \supset \overline{\upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)} \quad \& \quad \overline{(Neg(\upsilon\ Gen\ r))\ \varepsilon\ Flg(\varkappa)}\Big].$$

Four members of conjunction in the antecedent of implication in last are assumptions[1]:

1) $recursive(\varkappa)$;	3) $r\ \varepsilon\ \varkappa$;
2) ${}_{\omega}consist(\varkappa)$;	4) $recursive(r)$.

These assumptions allow to derive a decidability of undecidable formulas; furthermore, we will prove that these formulas are provable. Before we propose the proof of this result, let us turn to five supplementary lemmas[2].

[1] It is obviously from the original proof of Gödel's First incompleteness theorem [Gödel 1931b (engl.), pp. 98-100] and also from the reconstruction of it that was made recently by author (the reconstruction will be published soon in *Philosophy of Science*, the journal of SB RAS).

[2] The proofs of *lemmas* and *the decidability theorem* are given with the help of the symbolization technique of metamathematical predicates that was suggested by Gödel

Lemma 1. *If a class $\varkappa$ of formulas is recursive, then for the given class sign r can be defined whether it belongs to the class $\varkappa$ or not.*

Proof.

1. $recursive(\varkappa) \equiv \forall r(r\ \varepsilon\ \varkappa\ \vee\ \overline{r\ \varepsilon\ \varkappa})$	definition
2. $\forall r(A(r)) \equiv\ A(r)$	logic rule
3. $\forall r(r\ \varepsilon\ \varkappa\ \vee\ \overline{r\ \varepsilon\ \varkappa}) \equiv\ r\ \varepsilon\ \varkappa\ \vee\ \overline{r\ \varepsilon\ \varkappa}$	2 by $A(r) = r\ \varepsilon\ \varkappa\ \vee\ \overline{r\ \varepsilon\ \varkappa}$
4. $recursive(\varkappa) \equiv\ r\ \varepsilon\ \varkappa\ \vee\ \overline{r\ \varepsilon\ \varkappa}$	transitivity of $\equiv$, 1, 3
5. $recursive(\varkappa) \supset\ r\ \varepsilon\ \varkappa\ \vee\ \overline{r\ \varepsilon\ \varkappa}$	$\equiv$-elim., 4

□

Lemma 2. *If the given class sign r belongs to a class $\varkappa$, then υ Gen r belongs to a class $Flg(\varkappa)$.*

Proof.

1. $r\ \varepsilon\ \varkappa$	assumption
2. $r\ \varepsilon\ \varkappa\ \supset\ r\ \varepsilon\ \varkappa\ \vee\ Ax(r)\ \vee\ (y, z\ \varepsilon\ Flg(\varkappa)\ \&\ Fl(r, y, z))$	logic rule
3. $r\ \varepsilon\ \varkappa\ \vee\ Ax(r)\ \vee\ (y, z\ \varepsilon\ Flg(\varkappa)\ \&\ Fl(r, y, z))$	*m.p.*, 2, 1
4. $r\ \varepsilon\ Flg(\varkappa) \equiv\ r\ \varepsilon\ \varkappa\ \vee\ Ax(r)\ \vee\ (y, z\ \varepsilon\ Flg(\varkappa)\ \&\ Fl(r, y, z))$	definition
5. $r\ \varepsilon\ \varkappa\ \vee\ Ax(r)\ \vee\ (y, z\ \varepsilon\ Flg(\varkappa)\ \&\ Fl(r, y, z)) \supset\ r\ \varepsilon\ Flg(\varkappa)$	$\equiv$-elim., 4

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in his system P. It is assumed that this system and definitions of following notions and relations are known by readers (see Gödel 1931b (engl.)):
1) $recursive(\varkappa) \equiv \forall x(x\ \varepsilon\ \varkappa\ \vee\ \overline{x\ \varepsilon\ \varkappa})$;
2) *a class sign* is a formula (a combination of signs) that has the form $a(b)$, where b is a sign of type 1 (i.e. a variable of the natural numbers) and a a sign of type 2 (i.e. a class of numbers); or has one of forms $\sim(a)$, $(a) \vee (b)$, $x\Pi(a)$, where x may be any variable;
3) $x\ \varepsilon\ Flg(\varkappa) \equiv\ x\ \varepsilon\ \varkappa\ \vee\ Ax(x)\ \vee\ (y, z\ \varepsilon\ Flg(\varkappa)\ \&\ Fl(x, y, z))$;
4) $x\ Gen\ y\ \equiv\ x\Pi(y)$;
5) $Sb(x^{\upsilon}_{y}) \equiv\ Subst\ a\binom{\upsilon}{b}$;
6) ${}_{\omega}consist(\varkappa) \equiv \overline{\exists a\Big(\forall n[Sb(a^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]\ \&\ [Neg(\upsilon\ Gen\ a)]\ \varepsilon\ Flg(\varkappa)\Big)}$;
7) $\forall R(recursive(R) \equiv decid(R))$;
8) $recursive(r) \equiv recursive(R)\ \&\ R \leftrightarrows r$ (where the sign '$\leftrightarrows$' means a relation of one-to-one correspondence between an arbitrary relation (class) R and its isomorphic relation sign (class sign) r);
9) $Sb\big(r^{\upsilon}_{Z(n)}\big) \equiv r\big(Z(n)\big)$;
10) $decid(R) \equiv\ \exists r\big(R \rightarrow Bew(r)\ \&\ \overline{R} \rightarrow Bew(Neg(r))\big)$,
and also it is known that with the help of 5^{th} definition and some substitutions from the scheme of axiom III.1 of the system P, — i.e., $\upsilon\Pi(a) \supset\ Subst\ a\binom{\upsilon}{c}$, — the axiom $\upsilon\Pi(r) \supset\ Sb(r^{\upsilon}_{Z(n)})$ turns out.

6. $r \ \varepsilon \ Flg(\varkappa)$	*m.p.*, 5, 3
7. $\upsilon \ Gen \ r \equiv \ r$	logic rule
8. $\upsilon \ Gen \ r \ \varepsilon \ Flg(\varkappa)$	change rule, 7, 6
9. $r \ \varepsilon \ \varkappa \ \supset \ \upsilon \ Gen \ r \ \varepsilon \ Flg(\varkappa)$	$\supset$-enter., ass. elim., 1

□

Lemma 3. *Suppose υ Gen r belongs to a class $Flg(\varkappa)$. Let $Sb(r^{\upsilon}_{Z(n)})$ be a formula that results from a class sign r by a substitution for its free variable υ by a numeral of the number n; then does not exist a number n, such that $Sb(r^{\upsilon}_{Z(n)})$ does not belong to the class $Flg(\varkappa)$.*

Proof.

1. $\upsilon \ Gen \ r \ \varepsilon \ Flg(\varkappa)$	assumption
2. $\upsilon \ Gen \ r \ \equiv \ \upsilon\Pi(r)$	definition
3. $\upsilon\Pi(r) \ \varepsilon \ Flg(\varkappa)$	change rule, 2, 1
4. $\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})$	axiom
5. $Ax\big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big)$	Def. №42, 38
6. $Ax\big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big) \ \supset$ $Ax\big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big) \ \vee \ \big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big) \ \varepsilon \ \varkappa \vee$ $\Big(y, z \ \varepsilon \ Flg(\varkappa) \ \& \ Fl\big(\big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big), y, z\big)\Big)$	logic rule
7. $Ax\big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big) \ \vee \ \big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big) \ \varepsilon \ \varkappa \vee$ $\Big(y, z \ \varepsilon \ Flg(\varkappa) \ \& \ Fl\big(\big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big), y, z\big)\Big)$	*m.p.*, 6, 5
8. $\big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big) \ \varepsilon \ Flg(\varkappa) \equiv$ $Ax\big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big) \ \vee \ \big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big) \ \varepsilon \ \varkappa \vee$ $\Big(y, z \ \varepsilon \ Flg(\varkappa) \ \& \ Fl\big(\big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big), y, z\big)\Big)$	definition
9. $Ax\big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big) \ \vee \ \big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big) \ \varepsilon \ \varkappa \vee$ $\Big(y, z \ \varepsilon \ Flg(\varkappa) \ \& \ Fl\big(\big(\upsilon\Pi(r) \supset \ Sb(r^{\upsilon}_{Z(n)})\big), y, z\big)\Big) \ \supset$ $\big(\upsilon\Pi(r) \supset Sb(r^{\upsilon}_{Z(n)})\big) \ \varepsilon \ Flg(\varkappa)$	$\equiv$-elim., 8
10. $\big(\upsilon\Pi(r) \supset Sb(r^{\upsilon}_{Z(n)})\big) \ \varepsilon \ Flg(\varkappa)$	*m.p.*, 9, 7
11. $Fl\ \Big(Sb(r^{\upsilon}_{Z(n)}), \ \big(\upsilon\Pi(r) \supset Sb(r^{\upsilon}_{Z(n)})\big), \ \upsilon\Pi(r)\Big)$	Def. №43

12. $\upsilon\mathit{\Pi}(r)\ \varepsilon\ Flg(\varkappa)\ \&\ \big(\upsilon\mathit{\Pi}(r) \supset Sb(r^{\upsilon}_{Z(n)})\big)\ \varepsilon\ Flg(\varkappa)\ \&$ &-enter., 3,
$Fl\Big(Sb(r^{\upsilon}_{Z(n)}),\ \big(\upsilon\mathit{\Pi}(r) \supset Sb(r^{\upsilon}_{Z(n)})\big),\ \upsilon\mathit{\Pi}(r)\Big)$ 10, 11

13. $\upsilon\mathit{\Pi}(r)\ \varepsilon\ Flg(\varkappa)\ \&\ \big(\upsilon\mathit{\Pi}(r) \supset Sb(r^{\upsilon}_{Z(n)})\big)\ \varepsilon\ Flg(\varkappa)\ \&$ logic rule
$Fl\Big(Sb(r^{\upsilon}_{Z(n)}),\ \big(\upsilon\mathit{\Pi}(r) \supset Sb(r^{\upsilon}_{Z(n)})\big),\ \upsilon\mathit{\Pi}(r)\Big) \supset$
$\Big\{\upsilon\mathit{\Pi}(r)\ \varepsilon\ Flg(\varkappa)\ \&\ \big(\upsilon\mathit{\Pi}(r) \supset Sb(r^{\upsilon}_{Z(n)})\big)\ \varepsilon\ Flg(\varkappa)\ \&$
$Fl\Big(Sb(r^{\upsilon}_{Z(n)}),\ \big(\upsilon\mathit{\Pi}(r) \supset Sb(r^{\upsilon}_{Z(n)})\big),\ \upsilon\mathit{\Pi}(r)\Big)\Big\} \vee$
$Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ \varkappa\ \vee\ Ax\big(Sb(r^{\upsilon}_{Z(n)})\big)$

14. $\Big\{\upsilon\mathit{\Pi}(r)\ \varepsilon\ Flg(\varkappa)\ \&\ \big(\upsilon\mathit{\Pi}(r) \supset Sb(r^{\upsilon}_{Z(n)})\big)\ \varepsilon\ Flg(\varkappa)\ \&$ *m.p.*, 13, 12
$Fl\Big(Sb(r^{\upsilon}_{Z(n)}),\ \big(\upsilon\mathit{\Pi}(r) \supset Sb(r^{\upsilon}_{Z(n)})\big),\ \upsilon\mathit{\Pi}(r)\Big)\Big\} \vee$
$Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ \varkappa\ \vee\ Ax\big(Sb(r^{\upsilon}_{Z(n)})\big)$

15. $Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa) \equiv$ definition
$\Big\{\upsilon\mathit{\Pi}(r)\ \varepsilon\ Flg(\varkappa)\ \&\ \big(\upsilon\mathit{\Pi}(r) \supset Sb(r^{\upsilon}_{Z(n)})\big)\ \varepsilon\ Flg(\varkappa)\ \&$
$Fl\Big(Sb(r^{\upsilon}_{Z(n)}),\ \big(\upsilon\mathit{\Pi}(r) \supset Sb(r^{\upsilon}_{Z(n)})\big),\ \upsilon\mathit{\Pi}(r)\Big)\Big\} \vee$
$Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ \varkappa\ \vee\ Ax\big(Sb(r^{\upsilon}_{Z(n)})\big)$

16. $\Big\{\upsilon\mathit{\Pi}(r)\ \varepsilon\ Flg(\varkappa)\ \&\ \big(\upsilon\mathit{\Pi}(r) \supset Sb(r^{\upsilon}_{Z(n)})\big)\ \varepsilon\ Flg(\varkappa)\ \&$ ≡-elim., 15
$Fl\Big(Sb(r^{\upsilon}_{Z(n)}),\ \big(\upsilon\mathit{\Pi}(r) \supset Sb(r^{\upsilon}_{Z(n)})\big),\ \upsilon\mathit{\Pi}(r)\Big)\Big\} \vee$
$Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ \varkappa\ \vee\ Ax\big(Sb(r^{\upsilon}_{Z(n)})\big)\ \supset$
$Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)$

17. $Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)$ *m.p.*, 16, 14

18. $\forall n\big[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)\big]$ ∀-enter., 17

19. $\forall n\big[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)\big] \equiv \overline{\exists n\overline{\big[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)\big]}}$ logic rule

20. $\forall n\big[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)\big] \supset \overline{\exists n\overline{\big[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)\big]}}$ ≡-elim., 19

21. $\overline{\exists n\overline{\big[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)\big]}}$ *m.p.*, 20, 18

22. $\upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa) \supset \overline{\exists n\overline{\big[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)\big]}}$ ⊃-enter.,
ass. elim., 1

□

Lemma 4. *Suppose a class $\varkappa$ of formulas is ω-consistent. Let $Sb(r^{\upsilon}_{Z(n)})$ be a formula that results from a class sign r by a substitution for its free variable υ by a numeral of the number n; then either $Neg(\upsilon\ Gen\ r)$ does not belong to a class $Flg(\varkappa)$ or exists a number n, such that $Sb(r^{\upsilon}_{Z(n)})$ does not belong to the class $Flg(\varkappa)$.*

Proof.

1. $_{\omega}consist(\varkappa)$ — assumption

2. $_{\omega}consist(\varkappa) \equiv \overline{\exists r \left(\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]\ \&\ [Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)\right)}$ — definition

3. $_{\omega}consist(\varkappa) \supset \overline{\exists r \left(\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]\ \&\ [Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)\right)}$ — $\equiv$-elim., 2

4. $\overline{\exists r \left(\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]\ \&\ [Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)\right)}$ — *m.p.*, 3, 1

5. $\overline{\exists r \left(\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]\ \&\ [Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)\right)} \equiv \forall r \overline{\left(\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]\ \&\ [Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)\right)}$ — logic rule

6. $\overline{\exists r \left(\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]\ \&\ [Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)\right)} \supset \forall r \overline{\left(\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]\ \&\ [Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)\right)}$ — $\equiv$-elim., 5

7. $\forall r \overline{\left(\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]\ \&\ [Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)\right)}$ — *m.p.*, 6, 4

8. $\overline{\left(\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]\ \&\ [Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)\right)}$ — $\forall$-elim., 7

9. $\overline{\left(\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]\ \&\ [Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)\right)} \equiv \overline{\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]} \vee \overline{[Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)}$ — A. de Morgan rule

10. $\overline{\left(\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]\ \&\ [Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)\right)} \supset \overline{\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]} \vee \overline{[Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)}$ — $\equiv$-elim., 9

11. $\overline{\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]} \vee \overline{[Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)}$ — *m.p.*, 10, 8

12. $\overline{\forall n[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]} \equiv \exists n \overline{[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]}$ — logic rule

13. $\exists n \overline{[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]} \vee \overline{[Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)}$ — change rule, 12, 11

14. ${}_\omega consist(\varkappa) \supset$ $\exists n\overline{[Sb(r^v_{Z(n)})\ \varepsilon\ Flg(\varkappa)]}\ \vee\ \overline{[Neg(v\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)}$ — $\supset$-enter., ass. elim., 1

□

Lemma 5. *Suppose a class sign r is recursive; then either v Gen r or Neg(v Gen r) belongs to a class Flg($\varkappa$).*

Proof.

1.	$recursive(r)$	assumption
2.	$recursive(r) \equiv recursive(R)\ \&\ R \leftrightarrows r$	definition
3.	$recursive(r) \supset recursive(R)\ \&\ R \leftrightarrows r$	$\equiv$-elim., 2
4.	$recursive(R)\ \&\ R \leftrightarrows r$	*m.p.*, 3, 1
5.	$recursive(R)\ \&\ R \leftrightarrows r \supset recursive(R)$	logic rule
6.	$recursive(R)\ \&\ R \leftrightarrows r \supset R \leftrightarrows r$	logic rule
7.	$recursive(R)$	*m.p.*, 5, 4
8.	$R \leftrightarrows r$	*m.p.*, 6, 4
9.	$\forall R(recursive(R) \equiv decid(R))$	definition[3]
10.	$recursive(R) \equiv decid(R)$	$\forall$-elim., 9
11.	$recursive(R) \supset decid(R)$	$\equiv$-elim., 10
12.	$decid(R)$	*m.p.*, 11, 7
13.	$decid(R) \equiv\ \exists r\big(R \to Bew(r)\ \&\ \overline{R} \to Bew(Neg(r))\big)$	definition
14.	$decid(R) \supset\ \exists r\big(R \to Bew(r)\ \&\ \overline{R} \to Bew(Neg(r))\big)$	$\equiv$-elim., 13
15.	$\exists r\big(R \to Bew(r)\ \&\ \overline{R} \to Bew(Neg(r))\big)$	*m.p.*, 14, 12
16.	$R \to Bew(r)\ \&\ \overline{R} \to Bew(Neg(r))$	$\exists$-elim., 15, 8
17.	$Neg(R) \equiv \overline{R}$	definition
18.	$R \to Bew(r)\ \&\ Neg(R) \to Bew(Neg(r))$	change rule, 17, 16
19.	$Neg(Bew(r)) \to Neg(R)\ \&$ $Neg(Bew(Neg(r))) \to Neg(Neg(R))$	twice contraposition, 18
20.	$R \to Bew(r)\ \&\ Neg(R) \to Bew(Neg(r))\ \&$ $Neg(Bew(r)) \to Neg(R)\ \&$ $Neg(Bew(Neg(r))) \to Neg(Neg(R))$	&-enter., 18, 19

[3] The corollary of Gödel's Theorem V. See Gödel 1931b (engl.), p. 100.

21.	$Neg(Bew(Neg(r))) \to Neg(Neg(R))$ & $R \to Bew(r)$ & $Neg(Bew(r)) \to Neg(R)$ & $Neg(R) \to Bew(Neg(r))$	commutativity of &, 20
22.	$\big(Neg(Bew(Neg(r))) \to Neg(Neg(R))$ & $R \to Bew(r)\big)$ & $\big(Neg(Bew(r)) \to Neg(R)$ & $Neg(R) \to Bew(Neg(r))\big)$	associativity of &, 21
23.	$Neg(Neg(R)) \equiv R$	logic rule
24.	$\big(Neg(Bew(Neg(r))) \to R$ & $R \to Bew(r)\big)$ & $\big(Neg(Bew(r)) \to Neg(R)$ & $Neg(R) \to Bew(Neg(r))\big)$	change rule, 23, 22
25.	$\big(Neg(Bew(Neg(r))) \to Bew(r)\big)$ & $\big(Neg(Bew(r)) \to Bew(Neg(r))\big)$	twice transitivity of $\to$, 24
26.	$Neg(A) \to B \equiv A \vee B$	logic rule
27.	$Neg(Bew(Neg(r))) \to Bew(r) \equiv Bew(Neg(r)) \vee Bew(r)$	26 by $A = Bew(Neg(r))$, $B = Bew(r)$
28.	$Neg(Bew(r)) \to Bew(Neg(r)) \equiv Bew(r) \vee Bew(Neg(r))$	26 by $B = Bew(Neg(r))$, $A = Bew(r)$
29.	$\big(Bew(Neg(r)) \vee Bew(r)\big)$ & $\big(Bew(r) \vee Bew(Neg(r))\big)$	twice change rule, 27-28, 25
30.	$\big(Bew(Neg(r)) \vee Bew(r)\big)$ & $\big(Bew(r) \vee Bew(Neg(r))\big) \supset Bew(Neg(r)) \vee Bew(r)$	logic rule
31.	$Bew(Neg(r)) \vee Bew(r)$	*m.p.*, 30, 29
32.	$\upsilon\ Gen\ r \equiv\ r$	logic rule
33.	$Bew(Neg(\upsilon\ Gen\ r)) \vee Bew(\upsilon\ Gen\ r)$	change rule, 32, 31
34.	$\forall x[Bew(x) \to Bew_\varkappa(x)]$	statement (8)[4]
35.	$Bew(Neg(\upsilon\ Gen\ r)) \supset Bew_\varkappa(Neg(\upsilon\ Gen\ r))$	$\forall$-elim., 34
36.	$Bew(\upsilon\ Gen\ r) \supset Bew_\varkappa(\upsilon\ Gen\ r)$	$\forall$-elim., 34
37.	$\forall x[Bew_\varkappa(x) \equiv x\ \varepsilon\ Flg(\varkappa)]$	statement (7)[5]
38.	$Bew_\varkappa(Neg(\upsilon\ Gen\ r)) \equiv Neg(\upsilon\ Gen\ r)\ \varepsilon\ Flg(\varkappa)$	$\forall$-elim., 37
39.	$Bew_\varkappa(Neg(\upsilon\ Gen\ r)) \supset Neg(\upsilon\ Gen\ r)\ \varepsilon\ Flg(\varkappa)$	$\equiv$-elim., 38
40.	$Bew_\varkappa(\upsilon\ Gen\ r) \equiv \upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)$	$\forall$-elim., 37
41.	$Bew_\varkappa(\upsilon\ Gen\ r) \supset \upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)$	$\equiv$-elim., 40

[4] Gödel 1931b (engl.), p. 99
[5] Ibidem.

42.	$Bew(Neg(\upsilon\ Gen\ r)) \supset Neg(\upsilon\ Gen\ r)\ \varepsilon\ Flg(\varkappa)$	transitivity of $\supset$, 35, 39
43.	$Bew(\upsilon\ Gen\ r) \supset \upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)$	transitivity of $\supset$, 36, 41
44.	$Bew(Neg(\upsilon\ Gen\ r)) \vee Bew(\upsilon\ Gen\ r)\ \&$ $Bew(Neg(\upsilon\ Gen\ r)) \supset Neg(\upsilon\ Gen\ r)\ \varepsilon\ Flg(\varkappa)\ \&$ $Bew(\upsilon\ Gen\ r) \supset \upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)$	&-enter., 33, 42-43
45.	$(A \vee B\ \&\ A \supset C\ \&\ B \supset D) \supset C \vee D$	logic rule
46.	$\Big(Bew(Neg(\upsilon\ Gen\ r)) \vee Bew(\upsilon\ Gen\ r)\ \&$ $Bew(Neg(\upsilon\ Gen\ r)) \supset Neg(\upsilon\ Gen\ r)\ \varepsilon\ Flg(\varkappa)\ \&$ $Bew(\upsilon\ Gen\ r) \supset \upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)\Big) \supset$ $Neg(\upsilon\ Gen\ r)\ \varepsilon\ Flg(\varkappa) \vee \upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)$	45 by $A =$ $Bew(Neg(\upsilon\ Gen\ r))$, $B = Bew(\upsilon\ Gen\ r)$, $C = Neg(\upsilon\ Gen\ r)\ \varepsilon$ $\varepsilon\ Flg(\varkappa)$, $D =$ $\upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)$
47.	$Neg(\upsilon\ Gen\ r)\ \varepsilon\ Flg(\varkappa) \vee \upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)$	*m.p.*, 46, 44
48.	$recursive(r) \supset$ $Neg(\upsilon\ Gen\ r)\ \varepsilon\ Flg(\varkappa)\ \vee\ \upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)$	$\supset$-enter., ass. elim., 1

□

We now had been approaching to the main goal of present paper. The main result about the decidability of undecidable propositions consists in following:

Theorem 2. *(Decidability theorem). For every ω-consistent recursive class $\varkappa$ of formulas, for all recursive class sings r υ Gen r belongs to $Flg(\varkappa)$ whereas $Neg(\upsilon\ Gen\ r)$ does not belong to $Flg(\varkappa)$ (where υ is the free variable of r).*

Let us express the theorem symbolically:

$$\forall\varkappa,\ \forall r\Big[\big(recursive(\varkappa)\ \ \&\ \ {}_{\omega}consist(\varkappa)\ \ \&\ \ r\ \varepsilon\ \varkappa\ \ \&\ \ recursive(r)\big) \supset \upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)\ \ \&\ \ \overline{(Neg(\upsilon\ Gen\ r))\ \varepsilon\ Flg(\varkappa)}\Big].$$

Proof.

1.	$recursive(\varkappa)$	assumption 1
2.	$recursive(\varkappa) \supset\ r\ \varepsilon\ \varkappa\ \vee\ \overline{r\ \varepsilon\ \varkappa}$	Lemma 1
3.	$r\ \varepsilon\ \varkappa\ \vee\ \overline{r\ \varepsilon\ \varkappa}$	*m.p.*, 2, 1
4.	$r\ \varepsilon\ \varkappa$	assumption 2

5.	$r\ \varepsilon\ \varkappa\ \&\ (r\ \varepsilon\ \varkappa\ \vee\ \overline{r\ \varepsilon\ \varkappa})$	&-enter., 4, 3
6.	$r\ \varepsilon\ \varkappa\ \&\ (r\ \varepsilon\ \varkappa\ \vee\ \overline{r\ \varepsilon\ \varkappa}) \equiv r\ \varepsilon\ \varkappa$	elimination rule
7.	$r\ \varepsilon\ \varkappa\ \&\ (r\ \varepsilon\ \varkappa\ \vee\ \overline{r\ \varepsilon\ \varkappa}) \supset r\ \varepsilon\ \varkappa$	≡-elim., 6
8.	$r\ \varepsilon\ \varkappa$	*m.p.*, 7, 5
9.	$r\ \varepsilon\ \varkappa\ \supset\ \upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)$	Lemma 2
10.	$\upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)$	*m.p.*, 9, 8
11.	$\upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa) \supset \overline{\exists n \overline{\left[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)\right]}}$	Lemma 3
12.	$\overline{\exists n \overline{\left[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)\right]}}$	*m.p.*, 11, 10
13.	${}_{\omega}consist(\varkappa)$	assumption 3
14.	${}_{\omega}consist(\varkappa) \supset$ $\exists n \overline{[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]}\ \vee\ \overline{[Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)}$	Lemma 4
15.	$\exists n \overline{[Sb(r^{\upsilon}_{Z(n)})\ \varepsilon\ Flg(\varkappa)]}\ \vee\ \overline{[Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)}$	*m.p.*, 14, 13
16.	$\overline{[Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)}$	resolution, 15, 12
17.	$recursive(r)$	assumption 4
18.	$recursive(r) \supset$ $Neg(\upsilon\ Gen\ r)\ \varepsilon\ Flg(\varkappa)\ \vee\ \upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)$	Lemma 5
19.	$Neg(\upsilon\ Gen\ r)\ \varepsilon\ Flg(\varkappa)\ \vee\ \upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)$	*m.p.*, 18, 17
20.	$\upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)$	resolution, 19, 16
21.	$\upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)\ \&\ \overline{[Neg(\upsilon\ Gen\ r)]\ \varepsilon\ Flg(\varkappa)}$	&-enter., 20, 16
22.	$recursive(\varkappa)\ \&\ {}_{\omega}consist(\varkappa)\ \&\ r\ \varepsilon\ \varkappa\ \&\ recursive(r)$	&-enter., 1, 4, 13, 17
23.	$\big(recursive(\varkappa)\ \&\ {}_{\omega}consist(\varkappa)\ \&\ r\ \varepsilon\ \varkappa\ \&\ recursive(r)\big) \supset$ $\upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)\ \&\ \overline{(Neg(\upsilon\ Gen\ r))\ \varepsilon\ Flg(\varkappa)}$	⊃-enter., ass.-s elim., 1–4, 22, 21
24.	$\forall\varkappa, \forall r\Big[\big(recursive(\varkappa)\ \&\ {}_{\omega}consist(\varkappa)\ \&\ r\ \varepsilon\ \varkappa\ \&\ recursive(r)\big) \supset$ $\upsilon\ Gen\ r\ \varepsilon\ Flg(\varkappa)\ \&\ \overline{(Neg(\upsilon\ Gen\ r))\ \varepsilon\ Flg(\varkappa)}\Big]$	twice ∀-enter., 23

□

Q. E. D.

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